

CROSSED PRODUCTS BY FINITE CYCLIC GROUP ACTIONS WITH THE TRACIAL ROKHLIN PROPERTY

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ABSTRACT. We define the tracial Rokhlin property for actions of finite cyclic groups on stably finite simple unital C^* -algebras. We prove that the crossed product of a stably finite simple unital C^* -algebra with tracial rank zero by an action with this property again has tracial rank zero. Under a kind of weak approximate innerness assumption and one other technical condition, we prove that if the action has the tracial Rokhlin property and the crossed product has tracial rank zero, then the original algebra has tracial rank zero. We give examples showing how the tracial Rokhlin property differs from the Rokhlin property of Izumi.

We use these results, together with work of Elliott-Evans and Kishimoto, to give an inductive proof that every simple higher dimensional noncommutative torus is an AT algebra. We further prove that the crossed product of every simple higher dimensional noncommutative torus by the flip is an AF algebra, and that the crossed products of irrational rotation algebras by the standard actions of \mathbf{Z}_3 , \mathbf{Z}_4 , and \mathbf{Z}_6 are simple AH algebras with real rank zero. In the case of \mathbf{Z}_4 , we recover Walters' result that the crossed product is AF for a dense G_δ -set of rotation numbers.

0. INTRODUCTION

A higher dimensional noncommutative torus is a generalization of the rotation algebra A_θ to more generators. See Section 5 for details. The most important result of this paper is that every simple higher dimensional noncommutative torus is an AT algebra, that is, a direct limit of finite direct sums of C^* -algebras of the form $C(S^1, M_n)$ for varying values of n . Elliott and Evans proved [15] that the irrational rotation algebras are AT algebras, and Boca [6] showed that “most” simple higher dimensional noncommutative toruses are AT algebras. See the introduction to Section 5 for more of the history.

Our proof is by induction on the number of generators. Every higher dimensional noncommutative torus can be written as a successive crossed product by \mathbf{Z} , and the proof of Corollary 6.6 of [35] uses an inductive argument which works whenever the intermediate crossed products are all simple. Our contribution is a method, involving crossed products by actions of finite cyclic groups which have what we call the tracial Rokhlin property, for replacing any simple higher dimensional noncommutative torus by one to which something close to this argument applies. Our proof depends heavily on H. Lin's classification theorem for simple C^* -algebras with tracial rank zero [42].

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Our original motivation for the tracial Rokhlin property is that it enables us to prove that every simple higher dimensional noncommutative torus is an AT algebra. In retrospect, however, the following motivation is perhaps better. In [29] and [30], Izumi has started an intensive study of finite group actions with the Rokhlin property, which here, to minimize confusion, we call the strict Rokhlin property. The strict Rokhlin property imposes severe restrictions on the relation between the K-theory of the original algebra, the action of the group on this K-theory, and the K-theory of the crossed product. See especially Section 3 of [30]. In particular, the results there show that none of the actions appearing in any of our main applications has the strict Rokhlin property. Accordingly, a less restrictive version of the strict Rokhlin property is needed. The definition of the tracial Rokhlin property is obtained from the definition of the strict Rokhlin property in roughly the same way that H. Lin's definition of a tracially AF C*-algebra [37] is obtained from the definition of an AF algebra. We note that a similar concept, for integer actions on AF algebras, has appeared in [9].

There are standard actions of \mathbf{Z}_2 , \mathbf{Z}_3 , \mathbf{Z}_4 , and \mathbf{Z}_6 on the rotation algebras, and the action of \mathbf{Z}_2 is defined on every higher dimensional noncommutative torus as well. See the introductions to Sections 8 and 10 for details and some history. When the algebra is simple, all these actions have the tracial Rokhlin property. As a consequence, we show that the crossed products are always simple AH algebras with no dimension growth and real rank zero. For the action of \mathbf{Z}_2 on an irrational rotation algebra, it was already known [10] that the crossed is AF. For a dense G_δ -set of rotation numbers θ , the computation of K-theory of the crossed product $C^*(\mathbf{Z}_4, A_\theta)$ implies that this algebra is AF. We thus recover the main result of [61]. That paper, however, gives no information on the structure of $C^*(\mathbf{Z}_4, A_\theta)$ for other irrational values of θ , while our results imply in particular that $C^*(\mathbf{Z}_4, A_\theta)$ has real rank zero and stable rank one for all irrational θ . Our results are completely new for the actions of \mathbf{Z}_3 and \mathbf{Z}_6 on the irrational rotation algebras. In the case of the flip on a simple higher dimensional noncommutative torus, the crossed product was already known to be AF in “most” cases [6]; we are able to compute the K-theory and show that the crossed product is always AF.

Roughly the first half of this paper is a direct line to the proof that every simple noncommutative torus is an AT algebra. We develop the theory of crossed products by actions with the tracial Rokhlin property just far enough for this application. We start the second half of the paper with a theorem, Theorem 8.2, which we find very useful for proving that an action has the tracial Rokhlin property. Then we consider the standard actions of \mathbf{Z}_3 , \mathbf{Z}_4 , and \mathbf{Z}_6 on the irrational rotation algebras, and the flip action of \mathbf{Z}_2 on a simple higher dimensional noncommutative torus. Finally, we return to the general theory, and in particular present several examples involving UHF algebras which illustrate the limits of the theorems.

We now give a brief outline of the sections. Further introductory material can be found in Sections 1, 5, 8, and 12. We start in Section 1 by introducing the tracial Rokhlin property for actions of finite cyclic groups. In Section 2, we prove that crossed products of C*-algebras with tracial rank zero by such actions again have tracial rank zero. In Section 4, we give some conditions under which the dual action of an action with the tracial Rokhlin property again has the tracial Rokhlin property, and we examine the fixed point algebra of an action with the tracial

Rokhlin property. Section 3 introduces the notion of a tracially approximately inner automorphism, which is needed in Section 4.

Then we turn to higher dimensional noncommutative toruses. Section 5 contains various preliminaries. In Section 6, we prove that if A is a simple noncommutative torus, then the automorphism which multiplies one of the standard unitary generators by $\exp(2\pi i/n)$ generates an action of \mathbf{Z}_n with the tracial Rokhlin property. In Section 7, we use this result, work of Kishimoto on crossed products by \mathbf{Z} , and H. Lin’s classification theorem, to construct an inductive proof that every simple noncommutative torus is an AT algebra.

In the next three sections, we consider crossed products of irrational rotation algebra and higher dimensional noncommutative toruses by actions of finite cyclic groups which have already been studied. In Section 8, we show that the noncommutative Fourier transform on an irrational rotation algebra generates an action of \mathbf{Z}_4 with the tracial Rokhlin property, and use this to show that the crossed product is always a simple AH algebra with real rank zero, and is AF for “many” rotation numbers. This section contains a very useful criterion for an action on a simple C^* -algebra with tracial rank zero with unique tracial state to have the tracial Rokhlin property, a criterion which does not mention any projections. In Section 9, we show that the crossed products of an irrational rotation algebra by the “standard” actions of \mathbf{Z}_3 and \mathbf{Z}_6 are simple AH algebras with real rank zero. Then in Section 10 we show that the crossed product of a simple higher dimensional noncommutative torus by the flip is always AF.

In the last three sections, we return to the general theory. Section 11 contains several results which were not proved earlier because they are not needed for our main applications. In particular, the crossed product an AF algebra by an action with the (strict) Rokhlin property is again AF. In Section 12, we give examples of actions on the 2^∞ UHF algebra which show that many of our results can’t be improved. Section 13 contains some interesting questions to which we do not know the answers.

We use the notation \mathbf{Z}_n for $\mathbf{Z}/n\mathbf{Z}$; the p -adic integers will not appear in this paper. If A is a C^* -algebra and $\alpha: A \rightarrow A$ is an automorphism such that $\alpha^n = \text{id}_A$, then we write $C^*(\mathbf{Z}_n, A, \alpha)$ for the crossed product of A by the action of \mathbf{Z}_n generated by α .

We write A_{sa} for the set of selfadjoint elements of a C^* -algebra A . We write $p \precsim q$ to mean that the projection p is Murray-von Neumann equivalent to a subprojection of q , and $p \sim q$ to mean that p is Murray-von Neumann equivalent to q . Also, $[a, b]$ denotes the additive commutator $ab - ba$.

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1. THE TRACIAL ROKHLIN PROPERTY

In this section we introduce the tracial Rokhlin property and several related properties. We observe several elementary relations and consequences, and we prove several useful equivalent formulations.

Definition 1.1. Let A be a stably finite simple unital C^* -algebra and let $\alpha \in \text{Aut}(A)$ satisfy $\alpha^n = \text{id}_A$. We say that the action of \mathbf{Z}_n generated by α has the *tracial Rokhlin property* if for every finite set $F \subset A$, every $\varepsilon > 0$, every $N \in \mathbf{N}$, and every nonzero positive element $x \in A$, there are mutually orthogonal projections $e_0, e_1, \dots, e_{n-1} \in A$ such that:

- (1) $\|\alpha(e_j) - e_{j+1}\| < \varepsilon$ for $0 \leq j \leq n-2$.
- (2) $\|e_j a - a e_j\| < \varepsilon$ for $0 \leq j \leq n-1$ and all $a \in F$.
- (3) With $e = \sum_{j=0}^{n-1} e_j$, the projection $1-e$ is Murray-von Neumann equivalent to a projection in the hereditary subalgebra of A generated by x .
- (4) For every j with $0 \leq j \leq n-1$, there are N mutually orthogonal projections $f_1, f_2, \dots, f_N \leq e_j$, each of which is Murray-von Neumann equivalent to the projection $1-e$ of (3).

In this definition, $e = 1$ is allowed, in which case conditions (3) and (4) are vacuous.

Convention 1.2. With the notation as in Definition 1.1, we always take $e_n = e_0$, and we will usually take $e = \sum_{j=0}^{n-1} e_j$.

We do not require that $\sum_{j=0}^{n-1} e_j = 1$, as Izumi does for the Rokhlin property in Definition 3.1 of [29]. The terminology is motivated by H. Lin’s “tracially AF” [37] and “tracial topological rank” [38], in whose definitions there is also a “small” leftover projection.

We recall Izumi’s definition, specialized to the case of a finite cyclic group, but to emphasize the difference, we call it the strict Rokhlin property here.

Definition 1.3. Let A be a unital C^* -algebra and let $\alpha \in \text{Aut}(A)$ satisfy $\alpha^n = \text{id}_A$. We say that the action of \mathbf{Z}_n generated by α has the *strict Rokhlin property* if for every finite set $F \subset A$, and every $\varepsilon > 0$, there are mutually orthogonal projections $e_0, e_1, \dots, e_{n-1} \in A$ such that:

- (1) $\|\alpha(e_j) - e_{j+1}\| < \varepsilon$ for $0 \leq j \leq n-2$.
- (2) $\|e_j a - a e_j\| < \varepsilon$ for $0 \leq j \leq n-1$ and all $a \in F$.
- (3) $\sum_{j=0}^{n-1} e_j = 1$.

Remark 1.4. If an action of \mathbf{Z}_n on a unital C^* -algebra A has the strict Rokhlin property, then it has the tracial Rokhlin property.

If α is approximately inner, requiring $\sum_{j=0}^{n-1} e_j = 1$ forces $[1_A] \in K_0(A)$ to be divisible by n , and therefore rules out many C^* -algebras of interest. In fact, the strict Rokhlin property imposes much more stringent conditions on the K-theory. Theorem 3.3 and Lemma 3.2(1) of [30] show that if a nontrivial finite group G acts on a simple unital C^* -algebra A in such a way that the induced action on $K_*(A)$ is trivial, and if one of $K_0(A)$ and $K_1(A)$ is a nonzero free abelian group, then α does not have the strict Rokhlin property. Theorem 3.3 and the discussion preceding Theorem 3.4 of [30] show that if in addition G is cyclic of order n , then the strict Rokhlin property implies that $K_*(A)$ is uniquely n -divisible. It follows that the

actions considered in the applications in this paper never have the strict Rokhlin property.

For the tracial Rokhlin property to be likely to hold, the C*-algebra must have a reasonable number of projections. For reference, we recall here the definition of the property that seems most relevant.

Definition 1.5. Let A be a C*-algebra. We say that A has *Property (SP)* if every nonzero hereditary subalgebra in A contains a nonzero projection.

Lemma 1.6. Let A be a stably finite simple unital C*-algebra, and let $\alpha \in \text{Aut}(A)$ be an automorphism which satisfies $\alpha^n = \text{id}_A$ and such that the action of \mathbf{Z}_n generated by α has the tracial Rokhlin property. Then A has Property (SP) or the action generated by α has the strict Rokhlin property.

Proof. If A does not have Property (SP), then there is a nonzero positive element $x \in A$ which generates a hereditary subalgebra which contains no nonzero projection. ■

Our definition also seems too weak for use with purely infinite simple C*-algebras. For example, Condition (3) of Definition 1.1 would then be vacuous, and Condition (4) would be automatically satisfied whenever $e \neq 0$. On the other hand, the strict Rokhlin property still seems too strong, since even in a unital purely infinite simple C*-algebra, [1] need not be divisible by n in $K_0(A)$.

For the tracial Rokhlin property as we have defined it to be useful, not only stable finiteness but also some condition on comparison of projections seems to be necessary. Although we will not make immediate use of them, we recall here two well known conditions of this type. Others will appear later.

Definition 1.7. Let A be a unital C*-algebra. We say that A has *cancellation of projections* if whenever $n \in \mathbf{N}$ and $p, q, e \in M_n(A)$ are projections such that e is orthogonal to both p and q , and $p + e \sim q + e$, then $p \sim q$.

Definition 1.8. Let A be a unital C*-algebra. We say that the *order on projections over A is determined by traces* if whenever $n \in \mathbf{N}$ and $p, q \in M_n(A)$ are projections such that $\tau(p) < \tau(q)$ for all tracial states τ on A , then $p \lesssim q$.

This is just Blackadar's Second Fundamental Comparability Question. See 1.3.1 in [2].

Finally, we do not attempt to formulate the appropriate definition for actions on C*-algebras which are not simple. At the very least, one should ask that, given an arbitrary nonzero element $a \in A$, one can in addition require $\|eae\| \geq \|a\| - \varepsilon$. See Definition 2.1 of [37]. Quite possibly one should impose other conditions as well.

It is convenient to have a formally stronger version of the tracial Rokhlin property, in which $\|\alpha(e_{n-1}) - e_0\| < \varepsilon$ as well, and in which the defect projection is α -invariant. We will use a simple case of the following lemma, which is stated more generally for later use. We note for emphasis: in it, δ depends on E , N , and ε , but not on φ or the particular projections p, e_0, e_1, \dots, e_N .

Lemma 1.9. Let E be a finite dimensional C*-algebra, let S be a complete system of matrix units for E , let $N \in \mathbf{N}$, and let $\varepsilon > 0$. Then there exists $\delta > 0$ such that whenever A is a unital C*-algebra, $p \in A$ is a projection, $\varphi: E \rightarrow pAp$ is a unital homomorphism, and $e_0, e_1, \dots, e_N \in A$ are projections with $\sum_{j=0}^N e_j = 1$ and such

that $\|e_j\varphi(s) - \varphi(s)e_j\| \leq \delta$ for all j and all $s \in S$, then there is a unitary $u \in A$ with $\|u - 1\| < \varepsilon$ such that $(ue_ju^*)\varphi(a) = \varphi(a)(ue_ju^*)$ for all j and all $a \in E$.

Proof. Replacing E by $E \oplus \mathbf{C}$ and φ by $(a, \lambda) \mapsto \varphi(a) + \lambda(1-p)$, we see that it suffices to prove the result under the additional restriction $p = 1$ (with a different value of ε).

Now apply Lemma 2.5.10 of [39] to all dimensions up to $\dim(E)$, finding $\delta > 0$ such that whenever B and C are subalgebras of a unital C^* -algebra A such that $\dim(B) \leq \dim(E)$ and such that B has a complete system of matrix units each of which has distance less than $N(N+1)\delta$ from C , then there is a unitary $u \in A$ with $\|u - 1\| < \varepsilon$ such that $u^*Bu \subset C$. Apply this with $B = \varphi(E)$ and $C = \bigoplus_{j=0}^N e_j A e_j$. We note that if $s \in S$ then, because $\|e_j\varphi(s) - \varphi(s)e_j\| \leq \delta$ for all j ,

$$\text{dist}(\varphi(s), C) \leq \left\| s - \sum_{j=0}^N e_j \varphi(s) e_j \right\| \leq \sum_{j \neq k} \|e_j \varphi(s) e_k\| < N(N+1)\delta,$$

and that every element of C commutes with every e_j . ■

Lemma 1.10. Let A be a stably finite simple unital C^* -algebra and let $\alpha \in \text{Aut}(A)$ satisfy $\alpha^n = \text{id}_A$. Then the action of \mathbf{Z}_n generated by α has the tracial Rokhlin property if and only if the conditions of Definition 1.1 are satisfied, except that (1) is replaced by:

- (1') $e = \sum_{j=0}^{n-1} e_j$ is α -invariant, and $\|\alpha(e_j) - e_{j+1}\| < \varepsilon$ for $0 \leq j \leq n-1$, where, following Convention 1.2, we take $e_n = e_0$.

The analogous statement holds for the strict Rokhlin property of Definition 1.3.

Proof. That (1') implies (1) is trivial. For the reverse direction, we first prove (1') without the condition on α -invariance of e . Apply Definition 1.1 with $\frac{1}{n}\varepsilon$ in place of ε . Then, using $\alpha^n = \text{id}_A$,

$$\|e_0 - \alpha(e_{n-1})\| = \|\alpha^{n-1}(e_0) - e_{n-1}\| \leq \sum_{j=0}^{n-2} \|\alpha^{n-j-2}(\alpha(e_j) - e_{j+1})\| < (n-1)\frac{1}{n}\varepsilon < \varepsilon.$$

Now we arrange for α -invariance of $\sum_{j=0}^{n-1} e_j$. Without loss of generality $\|a\| \leq 1$ for all $a \in F$.

Let

$$\delta_1 = \min \left(\frac{\varepsilon}{5}, \frac{1}{4}, \frac{1}{4n+1} \right).$$

Apply Lemma 1.9 with $E = \mathbf{C} \oplus \mathbf{C}$, with $N = n$, and with δ_1 in place of ε . Let δ_2 be the resulting value of δ . Then set

$$\delta_3 = \min \left(\delta_1, \frac{\delta_2}{4n^2}, \frac{1}{4n^2} \right).$$

Apply the version of (1') without α -invariance of e , and with δ_3 in place of ε . Let f_0, f_1, \dots, f_{n-1} be the resulting projections. Set $f = \sum_{j=0}^{n-1} f_j$. Then

$$\|\alpha(f) - f\| \leq \sum_{j=0}^{n-1} \|\alpha(f_j) - f_{j+1}\| < n\delta_3.$$

Inductively, we get $\|\alpha^k(f) - f\| \leq kn\delta_3$ for $k \in \mathbf{N}$. In particular, with $c = \frac{1}{n} \sum_{j=0}^{n-1} \alpha^j(f)$, we get

$$\|c - f\| \leq \frac{1}{n} \sum_{j=0}^{n-1} \|\alpha^j(f) - f\| < \frac{1}{n} \sum_{j=0}^{n-1} jn\delta_3 \leq n^2\delta_3.$$

Since $n^2\delta_3 < \frac{1}{2}$, the projection $e = \chi_{(\frac{1}{2}, \infty)}(c)$ is defined, is α -invariant (because c is), and satisfies $\|e - c\| \leq \|c - f\|$, so that $\|e - f\| \leq 2\|c - f\| < 2n^2\delta_3$. It follows that $\|f_j e - e f_j\| < 4n^2\delta_3 \leq \delta_2$ for $0 \leq j \leq n-1$. By the choice of δ_2 and following Lemma 1.9, with the homomorphism $\varphi: \mathbf{C} \oplus \mathbf{C} \rightarrow A$ being $\varphi(\lambda_0, \lambda_1) = \lambda_0 e + \lambda_1(1-e)$ and with $f_0, f_1, \dots, f_{n-1}, 1-f$ in place of e_0, e_1, \dots, e_N , there is a unitary $u \in A$ with $\|u - 1\| < \delta_1$ such that $e_j = u f_j u^*$ commutes with e for $0 \leq j \leq n-1$. We get $\|e_j - f_j\| < 2\delta_1$.

Since $\sum_{j=0}^{n-1} f_j = f$ we get

$$\left\| e - \sum_{j=0}^{n-1} e_j \right\| \leq \|e - f\| + \sum_{j=0}^{n-1} \|f_j - e_j\| < 2n^2\delta_3 + 2n\delta_1 < 1.$$

Since e commutes with $\sum_{j=0}^{n-1} e_j$, this implies that $e = \sum_{j=0}^{n-1} e_j$. Since e is unitarily equivalent to f , we now have Conditions (3) and (4) of Definition 1.1.

From $\|e_j - f_j\| < 2\delta_1$, we get

$$\|e_j a - a e_j\| \leq \|f_j a - a f_j\| + 2\|e_j - f_j\| < \delta_3 + 4\delta_1 \leq \varepsilon.$$

This is Condition (2) of Definition 1.1. Moreover, for $0 \leq j \leq n-1$ we get

$$\begin{aligned} \|\alpha(e_j) - e_{j+1}\| &\leq \|\alpha(f_j) - f_{j+1}\| + \|e_j - f_j\| + \|e_{j+1} - f_{j+1}\| \\ &< \delta_3 + 2\delta_1 + 2\delta_1 \leq 5\delta_1 \leq \varepsilon. \end{aligned}$$

This is the estimate in Condition (1') of the lemma, and finishes the proof of the part about the tracial Rokhlin property.

The proof for the strict Rokhlin property is the same as the first part of the proof for the tracial Rokhlin property, since in this case $e = 1$ is automatically α -invariant. ■

We finish this section by proving that crossed products by actions with the tracial Rokhlin property are still simple.

Lemma 1.11. Let A be a stably finite simple unital C^* -algebra, and let $\alpha \in \text{Aut}(A)$ be an automorphism which satisfies $\alpha^n = \text{id}_A$ and such that the action of \mathbf{Z}_n generated by α has the tracial Rokhlin property. Then α^k is outer for $1 \leq k \leq n-1$.

Proof. Let $1 \leq k \leq n-1$ and let $u \in A$ be unitary. We prove that $\alpha^k \neq \text{Ad}(u)$. Apply Definition 1.1 with $F = \{u\}$, with $\varepsilon = \frac{1}{2n}$, with $N = 1$, and with $x = 1$. Then for $1 \leq k \leq n-1$, we have

$$\|\alpha^k(e_0) - e_k\| \leq \sum_{j=0}^{k-1} \|\alpha^{k-j-1}(\alpha(e_j) - e_{j+1})\| < k\varepsilon < \frac{1}{2}.$$

Also $\|e_0 u - ue_0\| < \varepsilon < \frac{1}{2}$, whence $\|ue_0 u^* - e_0\| < \frac{1}{2}$. The choice $N = 1$ implies that $e_0 \neq 0$, so orthogonality of e_0 and e_k implies $\|e_k - e_0\| = 1$. It follows that

$$\|\alpha^k(e_0) - ue_0 u^*\| \geq \|e_k - e_0\| - \|\alpha^k(e_0) - e_k\| - \|ue_0 u^* - e_0\| > 0.$$

Therefore $\alpha^k \neq \text{Ad}(u)$. ■

Corollary 1.12. Let A be a stably finite simple unital C*-algebra, and let $\alpha \in \text{Aut}(A)$ be an automorphism which satisfies $\alpha^n = \text{id}_A$ and such that the action of \mathbf{Z}_n generated by α has the tracial Rokhlin property. Then $C^*(\mathbf{Z}_n, A, \alpha)$ is simple.

Proof. This follows from Lemma 1.11 and Theorem 3.1 of [33]. ■

2. CROSSED PRODUCTS BY ACTIONS ON C*-ALGEBRAS WITH TRACIAL RANK ZERO

In this section, we prove that the crossed product of a C*-algebra with tracial rank zero by an action with the tracial Rokhlin property again has tracial rank zero.

The following result gives the criterion we use for a simple C*-algebra to have tracial rank zero. Note that, by Theorem 7.1(a) of [38], tracial rank zero is the same as tracially AF in the sense of Definition 2.1 of [37].

Proposition 2.1. Let A be a simple separable unital C*-algebra. Then A has tracial rank zero in the sense of Definition 3.1 of [38] if and only if the following holds.

For every finite set $S \subset A$, every $\varepsilon > 0$, every nonzero positive element $x \in A$, and every $N \in \mathbf{N}$, there is a projection $p \in A$ and a finite dimensional unital subalgebra $E \subset pAp$ (that is, p is the identity of E) such that:

- (1) $\|pa - ap\| < \varepsilon$ for all $a \in S$.
- (2) For every $a \in S$ there exists $b \in E$ such that $\|pap - b\| < \varepsilon$.
- (3) $1 - p$ is Murray-von Neumann equivalent to a projection in \overline{xAx} .
- (4) There are N mutually orthogonal projections in pAp , each of which is Murray-von Neumann equivalent to $1 - p$.

Proof. It is immediate that tracial rank zero implies the condition in the proposition.

The proof of the other direction is similar to the proof of Proposition 3.8 of [37], and we refer to parts of that proof for some of the argument. However, there appears to be a gap there, so we give more detail at the relevant point in our argument.

According to Definition 2.1 of [37], we must prove that if S , ε , x , and N are as in the hypotheses, and if in addition $a_0 \in A$ is a given nonzero element, then p and E can in addition be chosen so that $\|pa_0p\| > \|a_0\| - \varepsilon$. Without loss of generality $\|a_0\| = 1$ and $\varepsilon < 1$.

As in the first paragraph of the proof of Proposition 3.8 of [37], we may assume that $A \not\cong M_n$ for any n , and we conclude that A is stably finite and has Property (SP).

Now let S , ε , x , and N be as in the hypotheses, and let $a_0 \in A$ satisfy $\|a_0\| = 1$. Define a continuous function $f: [0, \infty) \rightarrow [0, 1]$ by

$$f(t) = \begin{cases} 0 & 0 \leq t \leq 1 - \frac{1}{4}\varepsilon \\ 8\varepsilon^{-1} (t - 1 + \frac{1}{4}\varepsilon) & 1 - \frac{1}{4}\varepsilon \leq t \leq 1 - \frac{1}{4}\varepsilon \\ 1 & 1 - \frac{1}{8}\varepsilon \leq t \end{cases} .$$

Then $b = f(a_0^* a_0) \neq 0$, so there is a nonzero projection $q_0 \in \overline{bAb}$, and by Lemma 3.1 of [37] there is a nonzero projection $q \leq q_0$ such that q is Murray-von Neumann

equivalent to a projection in \overline{xAx} . Use Lemma 3.2 of [37] to choose nonzero mutually orthogonal projections $q_1, q_2 \leq q$ such that $q_1 \sim q_2$.

Choose $\varepsilon_0 > 0$ so small that whenever D is a unital C^* -algebra and $e, f \in D$ are projections such that $\|ef\| < \varepsilon_0$, then $f \lesssim 1 - e$. Choose $\varepsilon_1 > 0$ so small that whenever D is a unital C^* -algebra and $b \in D_{sa}$ satisfies $\|b^2 - b\| < \varepsilon_1$, then there is a projection $e \in D$ with $\|e - b\| < \min(\frac{1}{12}\varepsilon, \frac{1}{2}\varepsilon_0^2)$. Clearly $\varepsilon_1 < 1$. Set

$$\delta = \min\left(\frac{1}{12}\varepsilon, \frac{1}{2}\varepsilon_0^2, \frac{1}{5}\varepsilon_1\right).$$

Apply the hypotheses with $S \cup \{q, a_0\}$ in place of S , with δ in place of ε , with q_1 in place of x , and with $\max(N, 2)$ in place of N . Let p and $E \subset pAp$ be the resulting projection and finite dimensional subalgebra. We finish the proof by showing that $\|pa_0p\| > \|a_0\| - \varepsilon$.

If $\|pq\| < \varepsilon_0$ then $q \lesssim 1 - p$, whence

$$1 - p \lesssim q_1 \leq 1 - p \quad \text{and} \quad 1 - p \lesssim q_2 \leq 1 - p.$$

Since $q_1q_2 = 0$, this contradicts stable finiteness of A . So $\|pq\| \geq \varepsilon_0$, whence $\|pqp\| \geq \varepsilon_0^2$. By the choice of p and E , there is $b \in E$ with

$$\|b - pqp\| < \min\left(\frac{1}{12}\varepsilon, \frac{1}{2}\varepsilon_0^2, \frac{1}{5}\varepsilon_1\right).$$

We have

$$\|(pqp)^2 - pqp\| = \|pq(pq - qp)p\| < \frac{1}{5}\varepsilon_1,$$

so

$$\begin{aligned} \|b^2 - b\| &\leq \|(pqp)^2 - pqp\| + \|b\| \cdot \|b - pqp\| + \|b - pqp\| \cdot \|pqp\| + \|b - pqp\| \\ &< (4 + \frac{1}{5}\varepsilon_1) \frac{1}{5}\varepsilon_1 \leq \varepsilon_1. \end{aligned}$$

Therefore there is a projection $e \in E$ with $\|e - b\| < \min(\frac{1}{2}\varepsilon_0^2, \frac{1}{12}\varepsilon)$. This gives $\|e - pqp\| < \frac{1}{2}\varepsilon_0^2 + \frac{1}{2}\varepsilon_0^2 = \varepsilon_0^2$. Since $\|pqp\| \geq \varepsilon_0^2$, we have $e \neq 0$. Furthermore, $\|e - pqp\| < \frac{1}{6}\varepsilon$.

Similarly $\|(qpq)^2 - qpq\| < \frac{1}{5}\varepsilon_1 < \varepsilon_1$, so there is a projection $e_0 \in qAq$ such that $\|e_0 - qpq\| < \frac{1}{12}\varepsilon$. Since $\|pqp - qpq\| \leq 2\|pq - qp\| < \frac{1}{6}\varepsilon$, we get $\|e - e_0\| < \frac{1}{6}\varepsilon + \frac{1}{6}\varepsilon + \frac{1}{12}\varepsilon < \frac{1}{2}\varepsilon$. Because $\varepsilon < 1$ this implies $e \sim e_0$, so $e_0 \neq 0$. Also, since $\|(a_0^*a_0)^{1/2}\| = 1$ we immediately get

$$\|(a_0^*a_0)^{1/2}e(a_0^*a_0)^{1/2} - (a_0^*a_0)^{1/2}e_0(a_0^*a_0)^{1/2}\| < \frac{1}{2}\varepsilon.$$

Since $e_0 \neq 0$ and $e_0 \leq q \in \overline{bAb}$, the definition of b implies $\|e_0(a_0^*a_0)e_0\| \geq 1 - \frac{1}{4}\varepsilon$. Therefore

$$\begin{aligned} 1 &\leq \|[a_0^*a_0]^{1/2}e_0]^*[(a_0^*a_0)^{1/2}e_0]\| + \frac{1}{4}\varepsilon = \|(a_0^*a_0)^{1/2}e_0(a_0^*a_0)^{1/2}\| + \frac{1}{4}\varepsilon \\ &< \|(a_0^*a_0)^{1/2}e(a_0^*a_0)^{1/2}\| + \frac{1}{2}\varepsilon + \frac{1}{4}\varepsilon \leq \|(a_0^*a_0)^{1/2}p(a_0^*a_0)^{1/2}\| + \frac{1}{2}\varepsilon + \frac{1}{4}\varepsilon \\ &= \|p(a_0^*a_0)p\| + \frac{3}{4}\varepsilon. \end{aligned}$$

So $\|p(a_0^*a_0)p\| > 1 - \frac{3}{4}\varepsilon$.

Using $\|pa_0 - a_0p\| < \frac{1}{12}\varepsilon$, we now get

$$\|pa_0p\|^2 = \|(pa_0^*p)(pa_0p)\| \geq \|pa_0^*a_0p\| - \frac{1}{12}\varepsilon > 1 - \frac{3}{4}\varepsilon - \frac{1}{12}\varepsilon > 1 - \varepsilon.$$

So $\|pa_0p\| > \sqrt{1 - \varepsilon} \geq 1 - \varepsilon$, as desired. ■

We also recall the properties of simple unital C^* -algebras with tracial rank zero.

Theorem 2.2. (H. Lin.) Let A be a simple unital C^* -algebra with tracial rank zero. Then A has real rank zero, stable rank one, and cancellation of projections (Definition 1.7). Moreover, the order on projections over A is determined by traces (Definition 1.8).

Proof. In view of Theorem 7.1(a) of [38], real rank zero and stable rank one are Theorem 3.4 of [37], and cancellation of projections is Lemma 3.3 of [37]. That the order is determined by traces is Theorems 6.8 and 6.13 of [38]. ■

Lemma 2.3. Let A be a simple unital C^* -algebra, and let $\alpha \in \text{Aut}(A)$ be an automorphism which satisfies $\alpha^n = \text{id}_A$ and such that the action of \mathbf{Z}_n generated by α has the tracial Rokhlin property. Then for every finite set $F \subset A$, every finite dimensional subalgebra $E \subset A$, every $\varepsilon > 0$, every $N \in \mathbf{N}$, and every nonzero positive element $x \in A$, there are mutually orthogonal projections $e_0, e_1, \dots, e_{n-1} \in A$ and a unitary $v \in A$ such that:

- (1) $\|\alpha(e_j) - e_{j+1}\| < \varepsilon$ for $0 \leq j \leq n-1$, where, following Convention 1.2, we take $e_n = e_0$.
- (2) $\|e_j a - ae_j\| < \varepsilon$ for $0 \leq j \leq n-1$ and all $a \in F$.
- (3) With $e = \sum_{j=0}^{n-1} e_j$, the projection $1 - e$ is Murray-von Neumann equivalent to a projection in \overline{xAx} .
- (4) For every j with $0 \leq j \leq n-1$, there are N mutually orthogonal projections $f_1, f_2, \dots, f_N \leq e_j$, each of which is Murray-von Neumann equivalent to the projection $1 - e$ of (3).
- (5) $\|v - 1\| < \varepsilon$, and e_j commutes with all elements of vEv^* for $0 \leq j \leq n-1$.
- (6) $\alpha(e) = e$.

Proof. Let F , E , ε , N , and x be as in the hypotheses. Without loss of generality $\|a\| \leq 1$ for all $a \in F$. Let S be a complete system of matrix units for E . Apply Lemma 1.9 with this E , S , and ε , and with n in place of N , obtaining $\delta_0 > 0$. Define $\delta = \min(\varepsilon, \delta_0)$.

Apply Lemma 1.10 with $F \cup S$ in place of F , with δ in place of ε , and with N and x as given. Let e_0, e_1, \dots, e_{n-1} and $e = \sum_{j=0}^{n-1} e_j$ be the resulting projections. Note that $\alpha(e) = e$. Parts (3), (4), and (6) of the conclusion are immediate. Part (1) follows from $\|\alpha(e_j) - e_{j+1}\| < \delta \leq \varepsilon$ for $0 \leq j \leq n-1$. It also follows that for all $a \in F \cup S$ we have $\|e_j a - ae_j\| < \delta \leq \varepsilon$, which gives Part (2).

Since $\|e_j a - ae_j\| < \delta \leq \delta_0$ for $a \in S$, the choice of δ_0 from Lemma 1.9, taking the homomorphism φ there to be the inclusion of E in A , provides a unitary $v_0 \in A$ with $\|v_0 - 1\| < \varepsilon$ and such that $v_0 e_j v_0^*$ commutes with every element of E . We obtain Part (5) of the conclusion by taking $v = v_0^*$. ■

Lemma 2.4. Let A be a unital C^* -algebra and let $\alpha \in \text{Aut}(A)$ satisfy $\alpha^n = \text{id}_A$. Let $w \in A$ be a unitary such that

$$w\alpha(w)\alpha^2(w)\cdots\alpha^{n-1}(w) = 1.$$

Then the automorphism $\beta = \text{Ad}(w) \circ \alpha \in \text{Aut}(A)$ satisfies $\beta^n = \text{id}_A$. Moreover, letting $u \in C^*(\mathbf{Z}_n, A, \alpha)$ and $v \in C^*(\mathbf{Z}_n, A, \beta)$ be the canonical unitaries implementing the automorphisms α and β , there is an isomorphism

$$\varphi: C^*(\mathbf{Z}_n, A, \beta) \rightarrow C^*(\mathbf{Z}_n, A, \alpha)$$

such that $\varphi(a) = a$ for all $a \in A$ and $\|\varphi(v) - u\| = \|w - 1\|$.

Proof. As is implicit in the statement of the lemma, we identify A with its image in each of the crossed products.

That $\beta^n = \text{id}_A$ is easy to check.

The unitary wu is in $C^*(\mathbf{Z}_n, A, \alpha)$. For $a \in A$, we have $(wu)a(wu)^* = w\alpha(a)w^* = \beta(a)$, by the definition of β . Moreover, using $u^n = 1$ and $uau^* = \alpha(a)$ for $a \in A$, we get

$$\begin{aligned} (wu)^n &= w(uwu^{-1})(u^2wu^{-2}) \cdots (u^{n-1}wu^{-(n-1)})u^n \\ &= [w\alpha(w)\alpha^2(w) \cdots \alpha^{n-1}(w)]u^n = 1. \end{aligned}$$

The universal property of crossed products therefore provides a homomorphism

$$\varphi: C^*(\mathbf{Z}_n, A, \beta) \rightarrow C^*(\mathbf{Z}_n, A, \alpha)$$

such that $\varphi(a) = a$ for all $a \in A$ and $\varphi(v) = wu$.

A similar argument shows that there is a homomorphism

$$\psi: C^*(\mathbf{Z}_n, A, \alpha) \rightarrow C^*(\mathbf{Z}_n, A, \beta)$$

such that $\psi(a) = a$ for all $a \in A$ and $\varphi(u) = w^*v$. One checks that $\psi \circ \varphi(b) = b$ for all $b \in C^*(\mathbf{Z}_n, A, \beta)$ by checking this for $b \in A$ and for $b = v$. Similarly $\varphi \circ \psi = \text{id}_{C^*(\mathbf{Z}_n, A, \alpha)}$. Therefore φ is an isomorphism.

Finally, $\|\varphi(v) - u\| = \|wu - u\| = \|w - 1\|$, as desired. ■

Lemma 2.5. Let A be a simple C*-algebra with Property (SP), and let $\alpha \in \text{Aut}(A)$ satisfy $\alpha^n = \text{id}_A$. Let $p \in A$ be a nonzero projection. Then there exists a nonzero projection $q \in A$ such that $\alpha^j(q) \precsim p$ for all j .

Proof. Using Property (SP) and Lemma 3.1 of [37], find a nonzero projection $e_1 \leq p$ such that $e_1 \precsim \alpha(p)$. In the same way, find a nonzero projection $e_2 \leq e_1$ such that $e_2 \precsim \alpha^2(p)$. Proceed inductively. Set $q = e_{n-1}$. Then q is a nonzero projection such that $q \precsim \alpha^j(p)$ for $0 \leq j \leq n-1$, hence for all j . ■

Lemma 2.6. Let A be a unital C*-algebra, let $\alpha \in \text{Aut}(A)$ satisfy $\alpha^n = \text{id}_A$, and let $\varepsilon > 0$. Let $e_0, e_1, \dots, e_{n-1} \in A$ be mutually orthogonal projections, let $e = \sum_{j=1}^{n-1} e_j$, and assume that $\alpha(e) = e$. Let $w_1, w_2, \dots, w_{n-1} \in A$ be partial isometries, satisfying

$$w_j w_j^* = e_j, \quad w_j^* w_j = \alpha(e_{j-1}), \quad \text{and} \quad \|w_j - e_j\| < \varepsilon$$

for $1 \leq j \leq n-1$. Then the element

$$w = 1 - e + w_1 + w_2 + \cdots + w_{n-1} + \alpha^{n-1}(w_1^*)\alpha^{n-2}(w_2^*) \cdots \alpha(w_{n-1}^*)$$

is a unitary in A satisfying:

- (1) $\|w - 1\| < 2n^2\varepsilon$.
- (2) $w\alpha(w)\alpha^2(w) \cdots \alpha^{n-1}(w) = 1$.
- (3) Following Convention 1.2, $w\alpha(e_{j-1})w^* = e_j$ for $1 \leq j \leq n$.
- (4) $\|(\text{Ad}(w) \circ \alpha)^k(a) - \alpha^k(a)\| \leq 4kn^2\varepsilon\|a\|$ for $a \in A$ and $k \in \mathbf{N}$.
- (5) $(\text{Ad}(w) \circ \alpha)^n = \text{id}_A$.

Proof. Define

$$z = \alpha^{n-1}(w_1^*)\alpha^{n-2}(w_2^*) \cdots \alpha(w_{n-1}^*).$$

We claim that

$$zz^* = e_0, \quad z^*z = \alpha(e_{n-1}), \quad \text{and} \quad \|z - e_0\| < n^2\varepsilon.$$

For the first, we observe that

$$\alpha(w_{n-1}^*)\alpha(w_{n-1}) = \alpha^2(e_{n-2}),$$

$$\alpha^2(w_{n-2}^*)\alpha(w_{n-1}^*)\alpha(w_{n-1})\alpha^2(w_{n-2}) = \alpha^2(w_{n-2}^*e_{n-2}w_{n-2}) = \alpha^3(e_{n-3}),$$

etc., ending with

$$zz^* = \alpha^n(e_0) = e_0.$$

For the second, we observe that

$$\alpha^{n-1}(w_1)\alpha^{n-1}(w_1^*) = \alpha^{n-1}(e_1),$$

$$\alpha^{n-2}(w_2)\alpha^{n-1}(w_1)\alpha^{n-1}(w_1^*)\alpha^{n-2}(w_2^*) = \alpha^{n-2}(w_2\alpha(e_1)w_2^*) = \alpha^{n-2}(e_2),$$

etc., ending with

$$z^*z = \alpha(e_{n-1}).$$

For the third, first observe that

$$\|\alpha^{n-k}(e_k) - e_0\| = \|\alpha^{-k}(e_k) - e_0\| = \|e_k - \alpha^k(e_0)\| < k\varepsilon,$$

whence

$$\|\alpha^{n-k}(w_k) - e_0\| \leq \|\alpha^{n-k}(w_k - e_k)\| + \|\alpha^{n-k}(e_k) - e_0\| < (k+1)\varepsilon.$$

Therefore

$$\|z - e_0\| < \sum_{k=1}^{n-1} (k+1)\varepsilon < n^2\varepsilon.$$

This completes the proof of the claim.

The element w of the statement is now defined by

$$w = 1 - e + w_1 + w_2 + \cdots + w_{n-1} + z.$$

Since $\alpha(e) = e$, we have $\sum_{j=0}^{n-1} \alpha(e_{j-1}) = e = \sum_{j=0}^{n-1} e_j$, so that w is in fact unitary. Clearly $w\alpha(e_{j-1})w^* = e_j$. This is Part (3) of the conclusion.

Part (1) of the conclusion is the estimate

$$\|w - 1\| \leq \|z - e_0\| + \sum_{j=1}^{n-1} \|w_j - e_j\| < n^2\varepsilon + (n-1)\varepsilon < 2n^2\varepsilon.$$

To prove Part (2), we simplify the notation by setting $w_0 = z$ and interpreting all subscripts as elements of \mathbf{Z}_n . From $w_j w_j^* = e_j$ and $w_j^* w_j = \alpha(e_{j-1})$ we get $\alpha^k(w_j)\alpha^k(w_j)^* = \alpha^k(e_j)$ and $\alpha^k(w_j)^*\alpha^k(w_j) = \alpha^{k+1}(e_{j-1})$. It follows that

$$w\alpha(w)\alpha^2(w)\cdots\alpha^{n-1}(w) = 1 - e + \sum_{j=0}^{n-1} w_j \alpha(w_{j-1})\alpha^2(w_{j-2})\cdots\alpha^{n-1}(w_{j-n+1}).$$

Because $\alpha^n = \text{id}_A$, we have

$$\begin{aligned} w_0 &= \alpha^{n-1}(w_1^*)\alpha^{n-2}(w_2^*)\cdots\alpha(w_{n-1}^*) \\ &= [\alpha^{-1}(w_1^*)\cdots\alpha^{-j}(w_j^*)][\alpha^{n-j-1}(w_{j-n+1}^*)\cdots\alpha(w_{-1}^*)], \end{aligned}$$

and we get, by substituting for the term $\alpha^j(w_0)$,

$$w_j \alpha(w_{j-1})\alpha^2(w_{j-2})\cdots\alpha^{n-1}(w_{j-n+1}) = e_j \alpha^n(e_{j-n}) = e_j^2 = e_j$$

for all j . This gives Part (2) of the conclusion.

Define $\beta = \text{Ad}(w) \circ \alpha \in \text{Aut}(A)$. Then $\beta^n = \text{id}_A$ by Part (2) and Lemma 2.4. This is Part (5). Also, for $a \in A$ we have

$$\|\beta(a) - \alpha(a)\| \leq 2\|w - 1\| \cdot \|a\| \leq 4n^2\varepsilon\|a\|,$$

and Part (4) of the conclusion follows by induction. ■

Theorem 2.7. Let A be a simple unital C*-algebra, and let $\alpha \in \text{Aut}(A)$ be an automorphism which satisfies $\alpha^n = \text{id}_A$ and such that the action of \mathbf{Z}_n generated by α has the tracial Rokhlin property. Suppose that A has tracial rank zero. Then $C^*(\mathbf{Z}_n, A, \alpha)$ has tracial rank zero.

Proof. It suffices to verify the condition of Proposition 2.1, for a finite set S of the form $S = F \cup \{u\}$, where F is a finite subset of the unit ball of A and $u \in C^*(\mathbf{Z}_n, A, \alpha)$ is the canonical unitary implementing the automorphism α . So let $F \subset A$ be a finite subset with $\|a\| \leq 1$ for all $a \in F$, let $\varepsilon > 0$, let $N \in \mathbf{N}$, and let $x \in C^*(\mathbf{Z}_n, A, \alpha)$ be a nonzero positive element.

The C*-algebra A has Property (SP) by Theorem 2.2. So we can apply Theorem 4.2 of [31], with $N = \{1\}$, to find a nonzero projection $p_0 \in \overline{A}$ which is Murray-von Neumann equivalent in $C^*(\mathbf{Z}_n, A, \alpha)$ to a projection in $\overline{x C^*(\mathbf{Z}_n, A, \alpha) x}$.

Since A has real rank zero (by Theorem 2.2), we may use Theorem 1.1(i) of [62] to find $(N+1)(n+1)$ nonzero Murray-von Neumann equivalent mutually orthogonal projections in $p_0 A p_0$. Call one of them p_1 . Use Lemma 2.5 to find a nonzero projection $p \leq p_1$ such that $\alpha^j(p) \lesssim p_1$ for all j .

Set

$$\varepsilon_0 = \frac{\varepsilon}{12(n+1)^5}.$$

Choose $\delta > 0$ with $\delta < \varepsilon_0$, and so small that whenever e and f are projections in a C*-algebra C such that $\|e - f\| < \delta$, then there is a partial isometry $s \in C$ such that

$$ss^* = e, \quad s^*s = f, \quad \text{and} \quad \|s - e\| < \varepsilon_0.$$

Apply the condition for tracial rank zero in Proposition 2.1 with δ in place of ε , with the finite set S there being

$$S_0 = F \cup \alpha(F) \cup \dots \cup \alpha^{n-1}(F),$$

and with p in place of a . We obtain a projection q_0 such that $1 - q_0 \lesssim p$, and a finite dimensional subalgebra E_0 with $q_0 \in E_0 \subset q_0 A q_0$, such that for every $a \in S_0$ we have $\|q_0 a - a q_0\| < \delta$ and there exists $b \in E_0$ such that $\|q_0 a q_0 - b\| < \delta$.

Apply Lemma 2.3 with S_0 in place of F , with $E_0 + \mathbf{C}(1 - q_0)$ in place of the finite dimensional subalgebra E , with δ in place of ε , with $N = 1$, and with p in place of x . We obtain a unitary $y \in A$ and mutually orthogonal projections $e_0, e_1, \dots, e_{n-1} \in A$ which commute with all elements of $y(E_0 + \mathbf{C}q_0)y^*$, such that $\|e_j a - a e_j\| < \delta$ for all $a \in S_0$, such that $\|y - 1\| < \delta$, such that $\|\alpha(e_j) - e_{j+1}\| < \delta$, and such that $e = \sum_{j=0}^{n-1} e_j$ is α -invariant and $1 - e \lesssim p$.

According to the choice of δ , for $1 \leq j \leq n-1$ there are partial isometries $w_j \in A$ such that

$$w_j w_j^* = e_j, \quad w_j^* w_j = \alpha(e_{j-1}), \quad \text{and} \quad \|w_j - e_j\| < \varepsilon_0.$$

Apply Lemma 2.6 to the e_j and w_j , with ε_0 in place of ε . We obtain a unitary w as there such that $\|w - 1\| < 2n^2\varepsilon_0$, and such that the automorphism $\beta = \text{Ad}(w) \circ \alpha$

satisfies $\beta^n = \text{id}_A$,

$$\|\beta^k(a) - \alpha^k(a)\| \leq 4kn^2\varepsilon_0\|a\|$$

for $k \in \mathbf{N}$ and $a \in A$, and $\beta(e_j) = e_{j+1}$ for all j .

Define

$$q = \sum_{k=0}^{n-1} \beta^k(e_0 y q_0 y^* e_0) \quad \text{and} \quad E = \bigoplus_{k=0}^{n-1} \beta^k(e_0 y E_0 y^* e_0).$$

By construction, e_0 commutes with $y q_0 y^*$, and the projections

$$e_0, \beta(e_0), \dots, \beta^{n-1}(e_0), 1 - e$$

are orthogonal, so that q is a β -invariant projection. Similarly, e_0 commutes with every element of $y E_0 y^*$, so E is a β -invariant finite dimensional subalgebra of A .

Let $a \in F$. We estimate $\|qa - aq\|$ and the distance from qaq to E . We begin by estimating $\|[\beta^k(e_0 y q_0 y^* e_0), a]\|$. Recall that $a \in F$ implies $\|a\| \leq 1$. Using

$$[e_0 q_0 e_0, \alpha^{n-k}(a)] = e_0 q_0 [e_0, \alpha^{n-k}(a)] + e_0 [q_0, \alpha^{n-k}(a)] e_0 + [e_0, \alpha^{n-k}(a)] q_0 e_0,$$

and because $\alpha^{n-k}(a) \in S_0$, we get

$$\|[e_0 q_0 e_0, \alpha^{n-k}(a)]\| \leq \|[q_0, \alpha^{n-k}(a)]\| + 2\|[e_0, \alpha^{n-k}(a)]\| < \delta + 2\delta = 3\delta.$$

So

$$\begin{aligned} \|[\beta^k(e_0 q_0 e_0), a]\| &= \|[e_0 q_0 e_0, \beta^{n-k}(a)]\| \\ &\leq 2\|\beta^{n-k}(a) - \alpha^{n-k}(a)\| + \|[e_0 q_0 e_0, \alpha^{n-k}(a)]\| \\ &< 8(n-k)n^2\varepsilon_0 + 3\delta < (8n^3 + 3)\varepsilon_0. \end{aligned}$$

Therefore

$$\begin{aligned} \|[\beta^k(e_0 y q_0 y^* e_0), a]\| &= \|[e_0 y q_0 y^* e_0, \beta^{n-k}(a)]\| \leq 4\|y - 1\| + \|[\beta^k(e_0 q_0 e_0), a]\| \\ &< 4\delta + (8n^3 + 3)\varepsilon_0 < (8n^3 + 7)\varepsilon_0. \end{aligned}$$

Now

$$\|[q, a]\| \leq \sum_{k=0}^{n-1} \|[\beta^k(e_0 y q_0 y^* e_0), a]\| < n(8n^3 + 7)\varepsilon_0 < \varepsilon.$$

We next estimate the distance from qaq to E . We begin by estimating

$$\begin{aligned} &\left\| qaq - \sum_{k=0}^{n-1} [\beta^k(e_0 y q_0 y^* e_0)] a [\beta^k(e_0 y q_0 y^* e_0)] \right\| \\ &\leq \sum_{k=0}^{n-1} \sum_{l \neq k} \|[\beta^k(e_0 y q_0 y^* e_0)] a [\beta^l(e_0 y q_0 y^* e_0)]\| \\ &\leq \sum_{k=0}^{n-1} \sum_{l \neq k} \left(\|[\beta^k(e_0 y q_0 y^* e_0)] [\beta^l(e_0 y q_0 y^* e_0)] a\| + \|[\beta^l(e_0 y q_0 y^* e_0), a]\| \right). \end{aligned}$$

In each summand in the last expression, the first term contains the expression $\beta^k(e_0) \beta^l(e_0) = e_k e_l$, which is zero because $k \neq l$. Therefore this term is zero. The second term was estimated above by $(8n^3 + 7)\varepsilon_0$, and there are fewer than n^2 summands, so we conclude that

$$\left\| qaq - \sum_{k=0}^{n-1} [\beta^k(e_0 y q_0 y^* e_0)] a [\beta^k(e_0 y q_0 y^* e_0)] \right\| < n^2(8n^3 + 7)\varepsilon_0.$$

By construction, there exists $b_k \in E_0$ such that $\|b_k - q_0\alpha^{n-k}(a)q_0\| < \delta$. Set $c_k = e_0y b_k y^* e_0$. Then set $c = \sum_{k=0}^{n-1} \beta^k(c_k) \in E$. Since e_0 commutes with yq_0y^* , we have, recalling at the fifth step that $\|\beta^k(a) - \alpha^k(a)\| \leq 4kn^2\varepsilon_0\|a\|$ and $\|y - 1\| < \delta$,

$$\begin{aligned} \| [e_0yq_0y^*e_0]\beta^{n-k}(a)[e_0yq_0y^*e_0] - c_k \| &= \| [e_0yq_0y^*]\beta^{n-k}(a)[yq_0y^*e_0] - e_0y b_k y^* e_0 \| \\ &\leq \|q_0y^*\beta^{n-k}(a)yq_0 - b_k\| \\ &\leq 2\|y - 1\| + \|q_0\beta^{n-k}(a)q_0 - b_k\| \\ &< 2\delta + 4(n-k)n^2\varepsilon_0 + \|q_0\alpha^{n-k}(a)q_0 - b_k\| \\ &< 2\delta + 4n^3\varepsilon_0 + \delta \leq (4n^3 + 3)\varepsilon_0. \end{aligned}$$

It follows that

$$\begin{aligned} &\left\| c - \sum_{k=0}^{n-1} \beta^k(e_0yq_0y^*e_0)a\beta^k(e_0yq_0y^*e_0) \right\| \\ &\leq \sum_{k=0}^{n-1} \|c_k - [e_0yq_0y^*e_0]\beta^{n-k}(a)[e_0yq_0y^*e_0]\| < n(4n^3 + 3)\varepsilon_0. \end{aligned}$$

Therefore

$$\|c - qaq\| < [n^2(8n^3 + 7) + n(4n^3 + 3)]\varepsilon_0 \leq 12(n+1)^5\varepsilon_0 < \varepsilon.$$

We regard q as a projection in $C^*(\mathbf{Z}_n, A, \beta)$. We further let

$$D = C^*(\mathbf{Z}_n, E, \beta|_E),$$

which is a finite dimensional subalgebra of $C^*(\mathbf{Z}_n, A, \beta)$. Let

$$\varphi: C^*(\mathbf{Z}_n, A, \beta) \rightarrow C^*(\mathbf{Z}_n, A, \alpha)$$

be the isomorphism of Lemma 2.4. We take the projection required in Lemma 2.1 to be q , and the finite dimensional subalgebra to be $\varphi(D)$. From what we just did, every element $a \in F$ satisfies $\|[q, a]\| < \varepsilon$ in $C^*(\mathbf{Z}_n, A, \beta)$, and there is $c \in E \subset D$ such that $\|c - qaq\| < \varepsilon$. Since $\varphi(q) = q$ and $\varphi(a) = a$ for all $a \in F$, in $C^*(\mathbf{Z}_n, A, \alpha)$ every element $a \in F$ satisfies $\|[q, a]\| < \varepsilon$, and there is $c \in E \subset \varphi(D)$ such that $\|c - qaq\| < \varepsilon$.

Letting $v \in C^*(\mathbf{Z}_n, A, \beta)$ be the canonical unitary implementing the automorphism β , we have $[q, v] = 0$, because $\beta(q) = q$, and $qvq \in D$. Therefore in $C^*(\mathbf{Z}_n, A, \alpha)$ we have $[q, \varphi(v)] = 0$ and $q\varphi(v)q \in \varphi(D)$. Lemma 2.4 gives

$$\|\varphi(v) - u\| = \|w - 1\| < 2n^2\varepsilon_0.$$

Therefore

$$\|[q, u]\| < 4n^2\varepsilon_0 < \varepsilon \quad \text{and} \quad \|quq - q\varphi(v)q\| < 2n^2\varepsilon_0 < \varepsilon.$$

We next show that $1 - q$ is Murray-von Neumann equivalent to a projection in $\overline{x}C^*(\mathbf{Z}_n, A, \alpha)x$. Recall that

$$1 - e \precsim p, \quad 1 - q_0 \precsim p, \quad \text{and} \quad (yq_0y^*)e_0 = e_0(yq_0y^*).$$

Furthermore,

$$1 - q = 1 - \sum_{k=0}^{n-1} \beta^k(e_0yq_0y^*e_0) = 1 - e + \sum_{k=0}^{n-1} \beta^k(e_0y[1 - q_0]y^*e_0).$$

Now, with the Murray-von Neumann equivalence in $C^*(\mathbf{Z}_n, A, \beta)$, we have

$$\beta^k(e_0y[1 - q_0]y^*e_0) \leq \beta^k(y[1 - q_0]y^*) \precsim 1 - q_0 \precsim p.$$

Because $C^*(\mathbf{Z}_n, A, \beta) \cong C^*(\mathbf{Z}_n, A, \alpha)$ via an isomorphism φ which fixes every element of A , we get $\beta^k(e_0y[1 - q_0]y^*e_0) \lesssim p$ in $C^*(\mathbf{Z}_n, A, \alpha)$ as well. Thus, in $C^*(\mathbf{Z}_n, A, \alpha)$, the projection $1 - q$ is the orthogonal sum of $n + 1$ projections, each of which is Murray-von Neumann equivalent to a subprojection of p . By the choice of p , there are $n + 1$ mutually orthogonal projections in $\overline{xC^*(\mathbf{Z}_n, A, \alpha)x}$, each Murray-von Neumann equivalent to p . Therefore $1 - q$ is Murray-von Neumann equivalent to a projection in $\overline{xC^*(\mathbf{Z}_n, A, \alpha)x}$.

It remains to prove Condition (4) of Proposition 2.1, that is, that there are N mutually orthogonal projections in $qC^*(\mathbf{Z}_n, A, \alpha)q$, each Murray-von Neumann equivalent in $C^*(\mathbf{Z}_n, A, \alpha)$ to $1 - q$. It suffices to prove this in A instead. Since A has cancellation of projections (by Theorem 2.2), it suffices to show that $N[1 - q] \leq [q]$ in $K_0(A)$; in fact, it suffices to show that $(N + 1)[1 - q] \leq [1]$. We saw in the previous paragraph that

$$1 - q = 1 - e + \sum_{k=0}^{n-1} \beta^k(e_0y[1 - q_0]y^*e_0).$$

By construction, we have $[1 - e] \leq [p] \leq [p_1]$ in $K_0(A)$. We also have

$$\begin{aligned} [\beta^k(e_0y[1 - q_0]y^*e_0)] &\leq [\beta^k(y[1 - q_0]y^*)] = [\alpha^k(y[1 - q_0]y^*)] \\ &= [\alpha^k(1 - q_0)] \leq [\alpha^k(p)] \leq [p_1]. \end{aligned}$$

Therefore $[1 - q] \leq (n + 1)[p_1]$. Since $(N + 1)(n + 1)[p_1] \leq [p_0] \leq [1]$, this gives $(N + 1)[1 - q] \leq [1]$, as desired. ■

3. TRACIALLY APPROXIMATELY INNER AUTOMORPHISMS

In this section, we introduce the notion of a tracially approximately inner automorphism. This condition is needed to prove that the dual action of an action with the tracial Rokhlin property again has the tracial Rokhlin property.

Here, we only prove those results of immediate use. Some further results are found in Section 11, and some examples are in Section 13.

Definition 3.1. Let A be a stably finite simple unital C^* -algebra and let $\alpha \in \text{Aut}(A)$. We say that α is *tracially approximately inner* if for every finite set $F \subset A$, every $\varepsilon > 0$, every $N \in \mathbb{N}$, and every nonzero positive element $x \in A$, there exist a projection $e \in A$ and a unitary $v \in eAe$ such that:

- (1) $\|\alpha(e) - e\| < \varepsilon$.
- (2) $\|ea - ae\| < \varepsilon$ for all $a \in F$.
- (3) $\|veaev^* - \alpha(eae)\| < \varepsilon$ for all $a \in F$.
- (4) $1 - e$ is Murray-von Neumann equivalent to a projection in \overline{xAx} .
- (5) There are N mutually orthogonal projections $f_1, f_2, \dots, f_N \leq e$, each of which is Murray-von Neumann equivalent to $1 - e$.

As in Definition 1.1, we allow $e = 1$, in which case conditions (4) and (5) are vacuous.

The motivation for the terminology is the same as that for the tracial Rokhlin property (Definition 1.1). As there, the condition does seem useful outside the stably finite case, so we include stable finiteness in the definition. As with the tracial Rokhlin property, we also do not attempt to formulate the correct version in the nonsimple case; at the very least such an extension should include the requirement that $\|ea_0e\| > \|a_0\| - \varepsilon$ for a predetermined nonzero $a_0 \in A$.

This condition seems to be appropriate for use with the tracial Rokhlin property. In particular, when we strengthen it in the presence of the tracial Rokhlin property below, we must allow $e \neq 1$ even if we start with an approximately inner automorphism.

Remark 3.2. Let A be a stably finite simple unital C^* -algebra and let $\alpha \in \text{Aut}(A)$. If α is approximately inner then α is tracially approximately inner. If α is tracially approximately inner and A does not have Property (SP), then α is approximately inner.

Example 12.3 shows that a tracially approximately inner automorphism need not be approximately inner, even on a simple AF algebra.

When A has cancellation of projections, the automorphism has finite order, and the action it generates has the tracial Rokhlin property, we can strengthen Condition (1) in Definition 3.1 to true invariance, and we can require that v have the same order as α . In Proposition 3.6 below, we will further strengthen this result, requiring for example $\alpha(v) = v$. We do not know whether cancellation of projections is really necessary.

Lemma 3.3. Let A be a stably finite simple unital C^* -algebra with cancellation of projections, and let $\alpha \in \text{Aut}(A)$ be tracially approximately inner and satisfy $\alpha^n = \text{id}_A$. Suppose that the action of \mathbf{Z}_n generated by α has the tracial Rokhlin property. Then for every finite set $F \subset A$, every $\varepsilon > 0$, every $N \in \mathbf{N}$, and every nonzero positive element $x \in A$, there exist a projection $e \in A$ and a unitary $v \in eAe$ such that:

- (1) $\alpha(e) = e$.
- (2) $\|ea - ae\| < \varepsilon$ for all $a \in F$.
- (3) $v^n = e$, and $\|veaev^* - \alpha(eae)\| < \varepsilon$ for all $a \in F$.
- (4) $1 - e$ is Murray-von Neumann equivalent to a projection in \overline{xAx} .
- (5) There are N mutually orthogonal projections $f_1, f_2, \dots, f_N \leq e$, each of which is Murray-von Neumann equivalent to $1 - e$.

The proof requires a fair amount of technical work to set up a rather short punch line, so we explain the basic idea before we begin. Assume for simplicity that F is α -invariant, that we can use the strict Rokhlin property to obtain projections e_0, e_1, \dots, e_{n-1} which exactly commute with every element of F and such that $\alpha(e_j) = e_{j-1}$ for $0 \leq j \leq n-1$ and $\sum_{j=0}^{n-1} e_j = 1$, and further that we can find a unitary $v_0 \in A$ such that $v_0av_0^* = \alpha(a)$ for $a \in F \cup \{e_0, e_1, \dots, e_{n-1}\}$. Then

$$v = e_1v_0e_0 + e_2v_0e_1 + \cdots + e_{n-1}v_0e_{n-2} + e_0v_0^{-(n-1)}e_{n-1}$$

is a unitary in A which satisfies $v^n = 1$ and $vav^* = \alpha(a)$ for all $a \in F$.

We also note that there is some relation between this proof and that of Lemma 3.3 of [29].

Proof of Lemma 3.3. Let $F \subset A$ be a finite set, let $\varepsilon > 0$, let $N \in \mathbf{N}$, and let $x \in A$ be a nonzero positive element.

Without loss of generality $n > 1$ and every $a \in F$ satisfies $\|a\| \leq 1$. Then $A \not\cong M_m$ for any m , because all automorphisms of M_m are inner and therefore don't have the tracial Rokhlin property. By Remark 3.2 and Lemma 1.6, there are two cases: either A has Property (SP) and $A \not\cong M_m$ for any m , or α is approximately

inner and has the strict Rokhlin property. We write the proof in the first case. In the second case, we take the projections

$$1 - f_0, p, g_1, g_2, \dots, g_{2N+2}$$

appearing below to be all zero, and we obtain the conclusion of the lemma with in addition $e = 1$.

Lemma 3.2 of [37] provides nonzero Murray-von Neumann equivalent orthogonal projections $g_1, g_2, \dots, g_{2N+2} \in \overline{xAx}$.

Define $\varepsilon_1 = \frac{1}{10}n^{-1}\varepsilon$. Choose $\varepsilon_2 > 0$ with

$$\varepsilon_2 \leq \min\left(\frac{\varepsilon_1}{4n+1}, \frac{\varepsilon_1}{2n(n+1)}\right),$$

and also so small that whenever D is a unital C*-algebra and $c \in D$ satisfies $\|cc^* - 1\| < 2n\varepsilon_2$ and $\|c^*c - 1\| < 2n\varepsilon_2$, then the unitary $u = c(c^*c)^{-1/2} \in D$ satisfies $\|u - c\| < \varepsilon_1$.

Apply Lemma 1.10 with $G_0 = \bigcup_{j=0}^{n-1} \alpha^j(F)$ in place of F , with $\frac{1}{5}\varepsilon_2$ in place of ε , with 1 in place of N , and with g_1 in place of x . Let

$$p_0^{(0)}, p_1^{(0)}, \dots, p_{n-1}^{(0)} \in A$$

be the resulting projections, and let $p = \sum_{j=0}^{n-1} p_j^{(0)}$. Note that $\alpha(p) = p$.

Apply Lemma 1.9 with $E = \mathbf{C}^n$, with $N = 1$, and with $\frac{1}{5}\varepsilon_2$ in place of ε . Let ε_3 be the resulting value of δ . Set

$$\varepsilon_4 = \min\left(\frac{1}{125}\varepsilon_2, \frac{1}{25}\varepsilon_3\right).$$

Apply Lemma 1.9 with $E = \mathbf{C}$, with $N = 1$, and with ε_4 in place of ε . Let ε_5 be the resulting value of δ . Choose $\varepsilon_6 > 0$ with

$$\varepsilon_6 \leq \min\left(\frac{\varepsilon_5}{4n+1}, \frac{\varepsilon_4}{n}, \frac{1}{2n}\right),$$

and also so small that whenever D is a unital C*-algebra and $p, q \in D$ are projections such that $\|p - q\| < 2n\varepsilon_6$, then there is a unitary $u \in D$ such that $upu^* = q$ and $\|u - 1\| < \varepsilon_4$.

Apply Definition 3.1 with

$$G = \{p, p_0^{(0)}, p_1^{(0)}, \dots, p_{n-1}^{(0)}\} \cup G_0$$

in place of F , with ε_6 in place of ε , with 1 in place of N , and with g_1 in place of x . Let f_0 be the resulting projection and let $w_0 \in f_0 A f_0$ be the resulting unitary.

Set

$$b = \frac{1}{n} \sum_{j=0}^{n-1} \alpha^j(f_0).$$

Then

$$\|b - f_0\| \leq \frac{1}{n} \sum_{k=0}^{n-1} \sum_{j=0}^{k-1} \|\alpha^j(\alpha(f_0) - f_0)\| < n\varepsilon_6 < \frac{1}{2}.$$

Therefore we can define $f_1 = \chi_{(\frac{1}{2}, \infty)}(b)$, and f_1 is a projection with $\alpha(f_1) = f_1$ and

$$\|f_1 - f_0\| \leq \|f_1 - b\| + \|b - f_0\| \leq 2\|b - f_0\| < 2n\varepsilon_6.$$

By the choice of ε_6 , there is a unitary $z_1 \in A$ such that $z_1 f_0 z_1^* = f_1$ and $\|z_1 - 1\| < \varepsilon_4$. We now have

$$\|f_1 p - p f_1\| \leq 2\|f_1 - f_0\| + \|f_0 p - p f_0\| < 4n\varepsilon_6 + \varepsilon_6 = (4n+1)\varepsilon_6 \leq \varepsilon_5.$$

By the choice of ε_5 using Lemma 1.9, and applying this lemma in the fixed point algebra A^α with $\varphi(\lambda) = \lambda f_1$ and p and $1-p$ in place of e_0 and e_1 , we obtain a unitary $z_2 \in A^\alpha$ such that $f = z_2 f_1 z_2^*$ commutes with p and $\|z_2 - 1\| < \varepsilon_4$. Note that $\alpha(f) = f$. Further define $w = z_2 z_1 w_0 (z_2 z_1)^*$. We now have

$$\|f - f_0\| \leq \|f - f_1\| + \|f_1 - f_0\| \leq 2\|z_2 - 1\| + \|f_1 - f_0\| < 2\varepsilon_4 + 2n\varepsilon_6 \leq 4\varepsilon_4.$$

and

$$\|w - w_0\| \leq 2\|z_2 z_1 - 1\| < 4\varepsilon_4.$$

Since $\varepsilon_6 \leq \varepsilon_4$, we now obtain the following in place of the conditions from Definition 3.1. (We give the proofs for (2) and (3) afterwards.)

- (1) $\alpha(f) = f$.
- (2) $\|fa - af\| < 25\varepsilon_4$ for all $a \in G$, and $fp = pf$.
- (3) The unitary $w \in fAf$ satisfies $\|wfafw^* - \alpha(faf)\| < 25\varepsilon_4$ for all $a \in G$.
- (4) $1 - f \lesssim g_1$.

For (2), we observe that, because $\|a\| \leq 1$,

$$\|fa - af\| \leq \|f_0 a - a f_0\| + 2\|f - f_0\| < \varepsilon_6 + 8\varepsilon_4 \leq 25\varepsilon_4.$$

For (3), we observe that

$$\begin{aligned} \|wfafw^* - \alpha(faf)\| &\leq \|w_0 f_0 a f_0 w_0^* - \alpha(f_0 a f_0)\| + 2\|w - w_0\| + 4\|f - f_0\| \\ &< \varepsilon_6 + 8\varepsilon_4 + 16\varepsilon_4 \leq 25\varepsilon_4. \end{aligned}$$

We use the choice of ε_3 from Lemma 1.9 and the inequality $25\varepsilon_4 \leq \varepsilon_3$, and apply this lemma in pAp with $\varphi: \mathbf{C}^n \rightarrow pAp$ being

$$\varphi(\lambda_0, \dots, \lambda_{n-1}) = \sum_{j=0}^{n-1} \lambda_j p_j^{(0)}$$

and with pf and $p(1-f)$ in place of e_0 and e_1 . We obtain a unitary $y_0 \in pAp$ such that the unitary $y = y_0 + 1 - p \in A$ has the property that $p_j = y p_j^{(0)} y^*$ commutes with f for $0 \leq j \leq n-1$, and that $\|y - 1\| < \frac{1}{5}\varepsilon_2$ and $\sum_{j=0}^{n-1} p_j = p$. Then $\|p_j - p_j^{(0)}\| < \frac{2}{5}\varepsilon_2$. From $\|ap_j^{(0)} - p_j^{(0)}a\| < \frac{1}{5}\varepsilon_2$ for $a \in G_0$, we now get $\|ap_j - p_j a\| < \varepsilon_2$, and from $\|\alpha(p_j^{(0)}) - p_{j+1}^{(0)}\| < \frac{1}{5}\varepsilon_2$ we get $\|\alpha(p_j) - p_{j+1}\| < \varepsilon_2$. Moreover,

$$\begin{aligned} \|wfp_j fw^* - \alpha(fp_j f)\| &\leq \|wfp_j^{(0)} fw^* - \alpha(fp_j^{(0)} f)\| + 2\|p_j - p_j^{(0)}\| \\ &< 25\varepsilon_4 + \frac{4}{5}\varepsilon_2 \leq \varepsilon_2. \end{aligned}$$

Therefore $\|wfp_j fw^* - fp_{j+1} f\| < 2\varepsilon_2$ for $0 \leq j \leq n-1$. Furthermore, for $a \in G_0$ we have

$$\|fp_j a - afp_j\| \leq \|f\| \cdot \|p_j a - ap_j\| + \|fa - af\| \cdot \|p_j\| < \varepsilon_2 + 25\varepsilon_4 \leq 2\varepsilon_2.$$

We also estimate

$$\begin{aligned} \|wfp - fpw\| &\leq \sum_{j=0}^{n-1} \|wfp_j - fp_{j+1}w\| \\ &= \sum_{j=0}^{n-1} \|wfp_j fw^* - fp_{j+1}f\| < 2n \cdot 2\varepsilon_2 = 4n\varepsilon_2. \end{aligned}$$

Using $w \in fAf$, for $a \in G_0$ we have

$$\begin{aligned} \|w(pf)a(pf)w^* - \alpha((pf)a(pf))\| &= \|wpfafpw^* - p\alpha(faf)p\| \\ &\leq \|p[wfafw^* - \alpha(faf)]p\| + 2\|wfp - fpw\| \\ &< 25\varepsilon_4 + 4n\varepsilon_2 \leq (4n+1)\varepsilon_2. \end{aligned}$$

Define $e = fp$ and $e_j = fp_j$. Then

$$e = \sum_{j=0}^{n-1} e_j \quad \text{and} \quad \|we - ew\| < 4n\varepsilon_2,$$

and for $0 \leq j \leq n-1$ and $a \in G_0$ we have

$$\begin{aligned} \|we_j w^* - e_{j+1}\| &< 2\varepsilon_2, \quad \|\alpha(e_j) - e_{j+1}\| < \varepsilon_2, \\ \|weaew^* - \alpha(eae)\| &< (4n+1)\varepsilon_2 \leq \varepsilon_1, \quad \text{and} \quad \|e_j a - ae_j\| < 2\varepsilon_2. \end{aligned}$$

Since p and f are α -invariant, so is e , which is Condition (1) of the conclusion. Also, if $a \in F$ then

$$\|ea - ae\| \leq \sum_{j=0}^{n-1} \|e_j a - ae_j\| < 2n\varepsilon_2 \leq \varepsilon_1 \leq \varepsilon,$$

so we have Condition (2) of the conclusion.

Following Convention 1.2, set $c = \sum_{j=0}^{n-1} e_{j+1}we_j \in eAe$. Then

$$\|c^*c - e\| \leq \sum_{j=0}^{n-1} \|e_j w^* e_{j+1} w e_j - e_j\| \leq \sum_{j=0}^{n-1} \|w^* e_{j+1} w - e_j\| < 2n\varepsilon_2.$$

Similarly $\|cc^* - e\| < 2n\varepsilon_2$. By the choice of ε_2 , the unitary $v_0 = c(c^*c)^{-1/2} \in eAe$ (functional calculus in eAe) satisfies $v_0 v_0^* = v_0^* v_0 = e$ and $\|v_0 - c\| < \varepsilon_1$. Also,

$$\begin{aligned} \|c - we\| &\leq \|we - ew\| + \|c - ew\| \\ &\leq \|we - ew\| + \sum_{j=0}^{n-1} \sum_{1 \leq k \leq n, k \neq j+1} \|e_k we_j\| \\ &< 4n\varepsilon_2 + n(n-1) \cdot 2\varepsilon_2 = 2n(n+1)\varepsilon_2 \leq \varepsilon_1. \end{aligned}$$

Therefore $\|v_0 - we\| < \varepsilon_1 + \varepsilon_1 = 2\varepsilon_1$. So, if $a \in G_0$ then

$$\|v_0 eaev_0^* - \alpha(eae)\| \leq 2\|v_0 - we\| + \|weaew^* - \alpha(eae)\| < 4\varepsilon_1 + \varepsilon_1 = 5\varepsilon_1.$$

Moreover, since c^*c commutes with all e_j , so does $(c^*c)^{-1/2}$. From the relation $c = \sum_{j=0}^{n-1} e_{j+1}ce_j$, we then get $v_0 = \sum_{j=0}^{n-1} e_{j+1}v_0e_j$. Since v_0 is unitary, this implies $v_0 e_j v_0^* = e_{j+1}$ for $0 \leq j \leq n-1$.

Now define

$$v = e_1 v_0 e_0 + e_2 v_0 e_1 + \cdots + e_{n-1} v_0 e_{n-2} + e_0 v_0^{-(n-1)} e_{n-1}.$$

Then v is a unitary in eAe such that $v^n = e$. Let $a \in F$. We need to estimate $\|veaev^* - \alpha(eae)\|$. Set $b = \sum_{j=0}^{n-1} e_j ae_j$. Then

$$\begin{aligned} \|b - eae\| &\leq \sum_{j=0}^{n-1} \sum_{0 \leq k \leq n-1, k \neq j} \|e_k ae_j\| \\ &\leq \sum_{j=0}^{n-1} \sum_{0 \leq k \leq n-1, k \neq j} \|e_k a - ae_k\| \cdot \|e_j\| < 2n(n-1)\varepsilon_2. \end{aligned}$$

Next, we calculate

$$\begin{aligned} v b v^* &= \sum_{j=0}^{n-1} v e_j a e_j v^* = v_0^{-(n-1)} e_{n-1} a e_{n-1} v_0^{n-1} + \sum_{j=0}^{n-2} v_0 e_j a e_j v_0^* \\ &= e_0 v_0^{-(n-1)} e a e v_0^{n-1} e_0 + \sum_{j=0}^{n-2} e_{j+1} v_0 e a e v_0^* e_{j+1}. \end{aligned}$$

Now $\|v_0 e a e v_0^* - \alpha(eae)\| < 5\varepsilon_1$ since $a \in G_0$. Also, since $\alpha^n = \text{id}_A$ it follows that all $\alpha^k(a)$, for $k \in \mathbf{Z}$, are in G_0 as well, so that an inductive argument gives $\|v_0^k \alpha(eae) v_0^{-k} - \alpha^{k+1}(eae)\| < 5k\varepsilon_1$ for $k \geq 1$. Putting $k = n-1$, using $\alpha^n = \text{id}_A$, and conjugating by $v_0^{-(n-1)}$, we obtain

$$\|\alpha(eae) - v_0^{-(n-1)} e a e v_0^{n-1}\| < 5(n-1)\varepsilon_1.$$

Therefore

$$\begin{aligned} &\left\| v b v^* - \sum_{j=0}^{n-1} e_j \alpha(eae) e_j \right\| \\ &\leq \|e_0 [v_0^{-(n-1)} e a e v_0^{n-1} - \alpha(eae)] e_0\| + \sum_{j=0}^{n-2} \|e_{j+1} [v_0 e a e v_0^* - \alpha(eae)] e_{j+1}\| \\ &< 5(n-1)\varepsilon_1 + (n-1) \cdot 5\varepsilon_1 = 10(n-1)\varepsilon_1. \end{aligned}$$

On the other hand,

$$\alpha(b) = \sum_{j=0}^{n-1} \alpha(e_j a e_j) = \sum_{j=0}^{n-1} \alpha(e_j) \alpha(eae) \alpha(e_j).$$

Therefore

$$\left\| \alpha(b) - \sum_{j=0}^{n-1} e_j \alpha(eae) e_j \right\| \leq \sum_{j=0}^{n-1} 2\|\alpha(e_j) - e_{j+1}\| < 2n\varepsilon_2 \leq \varepsilon_1.$$

Putting everything together, we get $\|v b v^* - \alpha(b)\| < (10n-9)\varepsilon_1$, so

$$\begin{aligned} \|veaev^* - \alpha(eae)\| &\leq \|v b v^* - \alpha(b)\| + 2\|b - eae\| \\ &< (10n-9)\varepsilon_1 + 4n(n-1)\varepsilon_2 \leq (10n-7)\varepsilon_1 < \varepsilon. \end{aligned}$$

This, together with the relation $v^n = 1$ from above, is Condition (3) of the conclusion.

To prove Condition (4) of the conclusion, write $1 - e = 1 - f + f(1 - p)$. By construction,

$$1 - f \lesssim g_1, \quad 1 - p \lesssim g_1, \quad \text{and} \quad g_1 \sim g_2.$$

Therefore $1 - e \lesssim g_1 + g_2 \in \overline{xAx}$, as required.

We prove Condition (5) of the conclusion. First note that

$$1 - f \lesssim g_1 \leq 1 - \sum_{j=3}^{2N+2} g_j \quad \text{and} \quad 1 - p \lesssim g_2 \leq 1 - \sum_{j=3}^{2N+2} g_j,$$

so

$$1 - e = 1 - f + f(1 - p) \lesssim g_1 + g_2 \leq 1 - \sum_{j=3}^{2N+2} g_j.$$

Therefore, because A has cancellation of projections, $\sum_{j=3}^{2N+2} g_j \lesssim e$. Choose a partial isometry $s \in A$ such that

$$s^*s = \sum_{j=3}^{2N+2} g_j \quad \text{and} \quad ss^* \leq e.$$

For $1 \leq j \leq N$ let $q_j^{(1)} \leq g_{j+2}$ be a projection with $q_j^{(1)} \sim 1 - f$ and let $q_j^{(2)} \leq g_{j+N+2}$ be a projection with $q_j^{(2)} \sim 1 - p$. Set $q_j = q_j^{(1)} + q_j^{(2)}$. Then the projections sq_js^* are N mutually orthogonal projections in eAe , each of which satisfies $1 - e \lesssim sq_js^*$. We have shown that there are N mutually orthogonal projections in eAe , each of which is Murray-von Neumann equivalent to $1 - e$, as desired. ■

Our further strengthening of Definition 3.1 requires two preliminary lemmas.

Lemma 3.4. Let D be a unital C*-algebra, let $n \in \mathbf{N}$, and let $\varepsilon > 0$. Then there is $\delta > 0$ such that whenever $v \in D$ is a unitary such that $v^n = 1$, whenever B is a unital C*-subalgebra of D , and whenever $c \in B$ satisfies $\|c - v\| < \delta$, then there is a unitary $w \in B$ such that $w^n = 1$ and $\|w - v\| < \varepsilon$.

Proof. This is semiprojectivity of \mathbf{C}^n , which is the universal C*-algebra generated by a unitary v with $v^n = 1$. (See Chapter 14 of [44].) ■

Lemma 3.5. Let D be a unital C*-algebra, let $n \in \mathbf{N}$, let $\alpha \in \text{Aut}(D)$ satisfy $\alpha^n = \text{id}_D$, and let $\varepsilon > 0$. Then there is $\delta > 0$ such that whenever $v \in D$ is a unitary such that $v^n = 1$, and whenever $c \in D$ satisfies $\|c - v\| < \delta$ and $\alpha(c) = \exp(-2\pi i/n)c$, then there is a unitary $w \in D$ such that

$$\|w - v\| < \varepsilon, \quad w^n = 1, \quad \text{and} \quad \alpha(w) = \exp(-2\pi i/n)w.$$

Proof. Let $\omega = \exp(2\pi i/n)$. Define an open set $U \subset S^1$ by

$$U = S^1 \setminus \{\exp(\pi i/n)\omega^j : 0 \leq j \leq n-1\}.$$

Let $f: U \rightarrow \mathbf{C}$ be the continuous function which takes the constant value ω^j on the open arc from $\exp(-\pi i/n)\omega^j$ to $\exp(\pi i/n)\omega^j$. Choose $\varepsilon_0 > 0$ with $\varepsilon_0 \leq \frac{1}{2}\varepsilon$ and so small that whenever $z \in D$ is a unitary such that $\|z^n - 1\| < \varepsilon_0$, then $\text{sp}(z) \subset U$ and $\|f(z) - z\| < \frac{1}{2}\varepsilon$. Choose $\delta > 0$ with $\delta \leq \frac{1}{2}\varepsilon$ and so small that whenever $v \in D$ is a unitary and whenever $c \in D$ satisfies $\|c - v\| < \delta$, then $\|c(c^*c)^{-1/2} - v\| < \frac{1}{n}\varepsilon_0$.

Now let $v \in D$ be a unitary such that $v^n = 1$, and let $c \in D$ satisfy $\|c - v\| < \delta$ and $\alpha(c) = \omega^{-1}c$. By the choice of δ , the unitary $z = c(c^*c)^{-1/2} \in D$ satisfies $\|z - v\| < \frac{1}{n}\varepsilon_0$. Therefore

$$\|z^n - 1\| = \|z^n - v^n\| \leq \sum_{k=1}^n \|z\|^{n-k} \cdot \|z - v\| \cdot \|v\|^{k-1} < n \cdot \left(\frac{1}{n}\varepsilon_0\right) = \varepsilon_0.$$

By the choice of ε_0 , the unitary $w = f(z)$ satisfies $\|w - z\| < \frac{1}{2}\varepsilon$. Therefore

$$\|w - v\| \leq \|w - z\| + \|z - v\| < \frac{1}{2}\varepsilon + \frac{1}{2n}\varepsilon \leq \varepsilon,$$

as desired.

It is clear that $w^n = 1$.

It remains only to show that $\alpha(w) = \omega^{-1}w$. Since $\alpha(c^*) = \omega c^*$, it follows that c^*c is in the fixed point algebra D^α . Therefore $\alpha(z) = \omega^{-1}z$. Since $f(\omega^{-1}\zeta) = \omega^{-1}f(\zeta)$ for every $\zeta \in U$, we get $f(\omega^{-1}z) = \omega^{-1}f(z)$. Therefore

$$\alpha(w) = \alpha(f(z)) = f(\alpha(z)) = f(\omega^{-1}z) = \omega^{-1}f(z) = \omega^{-1}w,$$

as desired. ■

Proposition 3.6. Let A be a stably finite simple unital C*-algebra with cancellation of projections, and let $\alpha \in \text{Aut}(A)$ be tracially approximately inner and satisfy $\alpha^n = \text{id}_A$. Suppose that the action of \mathbf{Z}_n generated by α has the tracial Rokhlin property. Then for every finite set $F \subset A$, every $\varepsilon > 0$, every $N \in \mathbf{N}$, and every nonzero positive element $x \in A$, there exist a projection $e \in A$ and unitaries $v_1, v_2 \in eAe$ such that the Conditions (1)–(5) of Lemma 3.3 are satisfied for both v_1 and v_2 in place of v , and in addition

$$\alpha(v_1) = v_1, \quad \alpha(v_2) = \exp(-2\pi i/n)v_2, \quad \text{and} \quad \|v_1v_2 - v_2v_1\| < \varepsilon.$$

As for Lemma 3.3, the proof requires a fair amount to technical work to set up a rather short punch line, and we explain the basic idea before we begin. Assume for simplicity that F is α -invariant, that we got from Lemma 3.3 a unitary $v_0 \in A$ such that $v_0^n = 1$ and $v_0av_0^* = \alpha(a)$ for all $a \in F$, and further that we can use the strict Rokhlin property to obtain projections e_0, e_1, \dots, e_{n-1} which exactly commute with v_0 and every element of F and such that $\alpha(e_j) = e_{j+1}$ for $0 \leq j \leq n-1$ and $\sum_{j=0}^{n-1} e_j = 1$. (We apply the approximate innerness condition and the Rokhlin property in the opposite order from the proof of Lemma 3.3.) Then

$$v_1 = \sum_{j=0}^{n-1} \alpha^j(e_0v_0e_0) \quad \text{and} \quad v_2 = \sum_{j=0}^{n-1} \exp(-2\pi ij/n)\alpha^j(e_0v_0e_0)$$

are unitaries in A which satisfy the conclusion.

Proof of Proposition 3.6. Let $F \subset A$ be a finite set, let $\varepsilon > 0$, let $N \in \mathbf{N}$, and let $x \in A$ be a nonzero positive element. Without loss of generality $\alpha(F) = F$ and $\|a\| \leq 1$ for all $a \in F$.

As in the proof of Lemma 3.3, there are two cases: either A has Property (SP) and $A \not\cong M_m$ for any m , or α is approximately inner and has the strict Rokhlin property. We again write the proof in the first case, and the second case is handled the same way as there.

Lemma 3.2 of [37] provides nonzero Murray-von Neumann equivalent orthogonal projections $g_1, g_2, \dots, g_{2N+2} \in \overline{xAx}$.

Choose $\varepsilon_1 > 0$ with $n^2\varepsilon_1 \leq \frac{1}{13}\varepsilon$ and so small that $2n\varepsilon_1$ will serve as δ in Lemmas 3.4 and 3.5 with the given value of n and with $\frac{1}{13}\varepsilon$ in place of ε . Choose $\varepsilon_2 > 0$ with $\varepsilon_2 \leq \min((n+1)^{-1}\varepsilon, \varepsilon_1)$ and so small that whenever D is a unital C*-algebra and $p, q \in D$ are projections such that $\|p - q\| < n\varepsilon_2$, then there is a unitary $u \in D$ such that $upu^* = q$ and $\|u - 1\| < \varepsilon_1$.

Apply Lemma 3.3 with F as given, with $\frac{1}{6}\varepsilon_2$ in place of ε , with 1 in place of N , and with g_1 in place of x . We obtain an α -invariant projection f_0 and a unitary $w_0 \in f_0 A f_0$ such that, in particular, $w_0^n = f_0$ and

$$\|af_0 - f_0 a\| < \frac{1}{6}\varepsilon_2 \quad \text{and} \quad \|w_0(f_0 a f_0) w_0^* - \alpha(f_0 a f_0)\| < \frac{1}{6}\varepsilon_2$$

for all $a \in F$.

We want to use Lemma 1.10 for the automorphism α , but we also want the resulting projections to commute with f_0 and w_0 , and we want to retain α -invariance of f_0 and the property $w_0^n = 1$. This requires perturbation both of the Rokhlin projections and of f_0 and w_0 . To this end, we apply Lemma 1.9 three times. The first time, we take $E = \mathbf{C}^n$ and use $n - 1$ in place of N and $\frac{1}{30}\varepsilon_2$ in place of ε . Let ρ be the resulting value of δ , and set $\varepsilon_3 = \min(\frac{1}{n}\rho, \frac{1}{30}\varepsilon_2)$. The second time, we take $E = \mathbf{C}^2$ and use $n - 1$ in place of N and $\frac{1}{5}\varepsilon_3$ in place of ε . Let ε_4 be the resulting value of δ , and also require $\varepsilon_4 \leq \frac{1}{5}\varepsilon_3$. The third time, we take $E = \mathbf{C}^2$ and use 1 in place of N and $\frac{1}{5}\varepsilon_4$ in place of ε . Let ε_5 be the resulting value of δ , and also require $\varepsilon_5 \leq \frac{1}{5}\varepsilon_4$.

Apply Lemma 1.10 to α , with $F \cup \{f_0, w_0\}$ in place of F , with $\frac{1}{n}\varepsilon_5$ in place of ε , with 1 in place of N , and with g_1 in place of x . We obtain mutually orthogonal projections $p_0^{(0)}, p_1^{(0)}, \dots, p_{n-1}^{(0)} \in A$. Set $p = \sum_{j=0}^{n-1} p_j^{(0)}$. Then in particular

$$\|p f_0 - f_0 p\| \leq \sum_{j=0}^{n-1} \|p_j^{(0)} f_0 - f_0 p_j^{(0)}\| < n \left(\frac{1}{n} \varepsilon_5 \right) = \varepsilon_5.$$

By the choice of ε_5 using Lemma 1.9, and applying this lemma in the fixed point algebra A^α with $\varphi(\lambda_1, \lambda_2) = \lambda_1 p + \lambda_2(1-p)$ and f_0 and $1 - f_0$ in place of e_0 and e_1 , we obtain a unitary $z_1 \in A^\alpha$ such that $f = z_1 f_0 z_1^*$ commutes with p and $\|z_1 - 1\| < \frac{1}{5}\varepsilon_4$. Note that $\alpha(f) = f$. Moreover, $\|f - f_0\| \leq 2\|z_1 - 1\|$, so

$$\begin{aligned} \|fp p_j^{(0)} - p_j^{(0)} fp\| &= \|(fp_j^{(0)} - p_j^{(0)} f)p\| \leq \|f_0 p_j^{(0)} - p_j^{(0)} f_0\| + 4\|z_1 - 1\| \\ &< \frac{1}{n}\varepsilon_5 + \frac{4}{5}\varepsilon_4 \leq \varepsilon_4. \end{aligned}$$

Also set $w_1 = z_1 w_0 z_1^* \in f A f$, so that $w_1^n = f$ and

$$\|w_1 - w_0\| \leq 2\|z_1 - 1\| < \frac{2}{5}\varepsilon_4.$$

By the choice of ε_4 using Lemma 1.9, and applying this lemma in the corner pAp with $\varphi(\lambda_1, \lambda_2) = \lambda_1 fp + \lambda_2(1-f)p$ and $p_0^{(0)}, p_1^{(0)}, \dots, p_{n-1}^{(0)}$ in place of e_0, e_1, \dots, e_N , we obtain a unitary $z_2 \in pAp$ such that $p_j = z_2 p_j^{(0)} z_2^*$ commutes with fp and $\|z_2 - p\| < \frac{1}{5}\varepsilon_3$. Because $p_j \leq p$ and f commutes with p , it now follows that p_j commutes with f for $0 \leq j \leq n - 1$. Moreover, $\|p_j - p_j^{(0)}\| \leq 2\|z_2 - p\|$, so

$$\begin{aligned} \|w_1 p_j - p_j w_1\| &\leq \|w_0 p_j^{(0)} - p_j^{(0)} w_0\| + 2\|p_j - p_j^{(0)}\| + 2\|w_1 - w_0\| \\ &< \varepsilon_5 + \frac{4}{5}\varepsilon_3 + \frac{4}{5}\varepsilon_4 \leq \frac{1}{25}\varepsilon_3 + \frac{4}{5}\varepsilon_3 + \frac{4}{25}\varepsilon_3 = \varepsilon_3. \end{aligned}$$

It follows that $\|w_1^k p_j - p_j w_1^k\| < k\varepsilon_3$ for $k \geq 1$. Therefore, with $\omega = \exp(2\pi i/n)$, the projections

$$q_l = \frac{1}{n} \sum_{k=0}^{n-1} \omega^{-lk} w_1^k$$

satisfy $\|q_l p_j - p_j q_l\| < n\varepsilon_3$. Since f commutes with w_1 and the p_j , we get $\|q_l f p_j - f p_j q_l\| < n\varepsilon_3$. Let $\varphi: \mathbf{C}^n \rightarrow f A f$ be the homomorphism which sends

$(1, \omega, \dots, \omega^{n-1})$ to w_1 . Then the q_l are the images under φ of the matrix units of \mathbf{C}^n . By the choice of ε_3 using Lemma 1.9, and applying this lemma in the corner fAf with this φ and with $fp_0, fp_1, \dots, fp_{n-1}$ in place of e_0, e_1, \dots, e_N , we obtain a unitary $z_3 \in fAf$ (which is u^* in the conclusion of the lemma) such that $w = z_3 w_1 z_3^*$ commutes with $fp_0, fp_1, \dots, fp_{n-1}$ and $\|z_3 - f\| < \frac{1}{30}\varepsilon_2$. Since $w \in fAf$ and $fp_j = p_j f$, we also get $wp_j = p_j w$.

We now have the algebraic relations

$$\alpha(f) = f, \quad w \in fAf, \quad w^n = f, \quad p = \sum_{j=0}^{n-1} p_j,$$

and, for $0 \leq j \leq n-1$,

$$p_j w = wp_j \quad \text{and} \quad p_j f = fp_j.$$

We further claim that if $a \in F$ then

$$\|fa - af\| < \varepsilon_2 \quad \text{and} \quad \|w(faf)w^* - \alpha(faf)\| < \varepsilon_2,$$

and, for $0 \leq j \leq n-1$,

$$\|p_j a - ap_j\| < \varepsilon_2 \quad \text{and} \quad \|\alpha(p_j) - p_{j+1}\| < \varepsilon_2.$$

We prove the claim. For $a \in F$, we have

$$\begin{aligned} \|fa - af\| &\leq 2\|f - f_0\| + \|f_0 a - af_0\| \leq 4\|z_1 - 1\| + \|f_0 a - af_0\| \\ &< \frac{4}{5}\varepsilon_4 + \frac{1}{6}\varepsilon_2 \leq \varepsilon_2. \end{aligned}$$

This is the first estimate. For the second,

$$\begin{aligned} \|w(faf)w^* - \alpha(faf)\| &\leq 2\|w - w_0\| + 4\|f - f_0\| + \|w_0(f_0 a f_0)w_0^* - \alpha(f_0 a f_0)\| \\ &< 4\|z_3 - f\| + 2\|w_1 - w_0\| + \frac{8}{5}\varepsilon_4 + \frac{1}{6}\varepsilon_2 \\ &< \frac{4}{30}\varepsilon_2 + \frac{12}{5}\varepsilon_4 + \frac{5}{6}\varepsilon_2 < \varepsilon_2. \end{aligned}$$

For the third,

$$\begin{aligned} \|p_j a - ap_j\| &\leq 2\|p_j - p_j^{(0)}\| + \|p_j^{(0)} a - ap_j^{(0)}\| \leq 4\|z_2 - p\| + \|p_j^{(0)} a - ap_j^{(0)}\| \\ &< \frac{4}{5}\varepsilon_3 + \frac{1}{n}\varepsilon_5 \leq \varepsilon_3 \leq \varepsilon_2. \end{aligned}$$

Finally,

$$\begin{aligned} \|\alpha(p_j) - p_{j+1}\| &\leq \|p_j - p_j^{(0)}\| + \|p_{j+1} - p_{j+1}^{(0)}\| + \|\alpha(p_j^{(0)}) - p_{j+1}^{(0)}\| \\ &\leq 4\|z_2 - p\| + \|\alpha(p_j^{(0)}) - p_{j+1}^{(0)}\| < \frac{4}{5}\varepsilon_3 + \frac{1}{n}\varepsilon_5 \leq \varepsilon_3 \leq \varepsilon_2. \end{aligned}$$

We now define $e = fp$. Then $\alpha(e) = e$, which is Condition (1) of the conclusion of Lemma 3.3. Furthermore, for $a \in F$ we have

$$\|ea - ae\| \leq \|f\| \sum_{j=0}^{n-1} \|p_j a - ap_j\| + \|fa - af\| \cdot \|p\| < (n+1)\varepsilon_2 \leq \varepsilon.$$

This is Condition (2) of the conclusion of Lemma 3.3.

Next, define $v_0 = we$, which is a unitary in eAe with $v_0^n = e$, and $e_j = ep_j$ for $0 \leq j \leq n-1$, which are projections in eAe which commute with v_0 . Since $\|\alpha(p_j) - p_{j+1}\| < \varepsilon_2$ and $\alpha(e) = e$, we get $\|\alpha^j(e_j) - e_{j+1}\| < \varepsilon_2$ for $0 \leq j \leq n-1$. So

$$\|\alpha^j(e_0) - e_j\| < j\varepsilon_2 \leq n\varepsilon_2$$

for $0 \leq j \leq n-1$. By the choice of ε_2 , there are unitaries $y_j \in eAe$ with $\|y_j - e\| < \varepsilon_1$ such that $y_j \alpha^j(e_0) y_j^* = e_j$ for $0 \leq j \leq n-1$. Define

$$c_1 = \sum_{j=0}^{n-1} y_j \alpha^j(e_0 v_0 e_0) y_j^* \quad \text{and} \quad d_1 = \sum_{j=0}^{n-1} \alpha^j(e_0 v_0 e_0).$$

Because the e_j are orthogonal, and because v_0 commutes with the e_j and satisfies $v_0^n = e$, it follows that c_1 is a unitary in eAe with $c_1^n = e$. Moreover, d_1 is α -invariant and satisfies

$$\|c_1 - d_1\| \leq \sum_{j=0}^{n-1} 2\|y_j - e\| < 2n\varepsilon_1.$$

By the choice of ε_1 using Lemma 3.4, there is a unitary v_1 in the fixed point algebra $(eAe)^\alpha$ such that $v_1^n = 1$ and $\|v_1 - c_1\| < \frac{1}{13}\varepsilon$. Further define

$$c_2 = \sum_{j=0}^{n-1} \omega^j y_j \alpha^j(e_0 v_0 e_0) y_j^* \quad \text{and} \quad d_2 = \sum_{j=0}^{n-1} \omega^j \alpha^j(e_0 v_0 e_0).$$

By the same argument as above, c_2 is a unitary in eAe with $c_2^n = e$, and

$$\alpha(d_2) = \omega^{-1} d_2 \quad \text{and} \quad \|c_2 - d_2\| < 2n\varepsilon_1.$$

So from Lemma 3.5 we get a unitary $v_2 \in eAe$ such that

$$v_2^n = 1, \quad \|v_2 - c_2\| < \frac{1}{13}\varepsilon, \quad \text{and} \quad \alpha(v_2) = \omega^{-1} v_2.$$

We also observe that $c_1 c_2 = c_2 c_1$. From $\|v_1 - c_1\| < \frac{1}{13}\varepsilon$ and $\|v_2 - c_2\| < \frac{1}{13}\varepsilon$ we therefore get $\|v_1 v_2 - v_2 v_1\| < \frac{4}{13}\varepsilon < \varepsilon$. We have proved the new conditions on both v_1 and v_2 .

We now estimate

$$\|v_1(eae)v_1^* - \alpha(eae)\| \quad \text{and} \quad \|v_2(eae)v_2^* - \alpha(eae)\|$$

for $a \in F$. We begin by observing that

$$\|e_j a - a e_j\| \leq \|fa - af\| \cdot \|p_j\| + \|f\| \cdot \|p_j a - ap_j\| < 2\varepsilon_2.$$

Then set $b = \sum_{j=0}^{n-1} e_j a e_j$. Since the e_j are orthogonal,

$$\|eae - b\| \leq \sum_{j=0}^{n-1} \sum_{0 \leq k \leq n-1, k \neq j} \|e_j a - a e_j\| < 2n(n-1)\varepsilon_2 \leq 2n^2\varepsilon_1.$$

Also,

$$\left\| \alpha(b) - \sum_{j=0}^{n-1} e_j \alpha(a) e_j \right\| \leq \sum_{j=0}^{n-1} 2\|e_{j+1} - \alpha(e_j)\| < 2n\varepsilon_2 \leq 2n\varepsilon_1.$$

The next step is to estimate $\left\| c_l b c_l^* - \sum_{j=0}^{n-1} e_j \alpha(a) e_j \right\|$. First, use $y_j \alpha^j(e_0) y_j^* = e_j$ and $e_0 v_0 = v_0 e_0$ to calculate

$$\alpha^j(e_0 v_0 e_0) y_j^* e_j = \alpha^j(e_0 v_0 e_0) \alpha^j(e_0) y_j^* e = \alpha^j(e_0 v_0) y_j^* e.$$

Also $\alpha(e) = e$ and

$$\|v_0 e a e v_0^* - \alpha(eae)\| = \|e w f a f w^* e - e \alpha(faf) e\| < \varepsilon_2.$$

Now $\alpha^{-j}(a) \in F$, so the above applies with $\alpha^j(a)$ in place of a . Using this fact at the last step, we get

$$\begin{aligned} & \| [y_j \alpha^j(e_0 v_0 e_0) y_j^*] e_j a e_j [y_j \alpha^j(e_0 v_0 e_0) y_j^*]^* - e_j \alpha(a) e_j \| \\ &= \| \alpha^j(e_0 v_0) y_j^* e a e y_j \alpha^j(v_0^* e_0) - y_j^* e_j \alpha(e a e) e_j y_j \| \\ &\leq \| y_j^* e a e y_j - e a e \| \\ &\quad + \| \alpha^j(e_0) \alpha^j(v_0 e \alpha^{-j}(a) e v_0^*) \alpha^j(e_0) - \alpha^j(e_0) y_j^* \alpha(e a e) y_j \alpha^j(e_0) \| \\ &\leq 2\|y_j - e\| + \|v_0 e \alpha^{-j}(a) e v_0^* - \alpha^{-j+1}(e a e)\| + \|\alpha(e a e) - y_j^* \alpha(e a e) y_j\| \\ &\leq 2\|y_j - e\| + \|v_0[e \alpha^{-j}(a) e] v_0^* - \alpha(e \alpha^{-j}(a) e)\| + 2\|y_j - e\| \\ &< 4\varepsilon_1 + \varepsilon_2 \leq 5\varepsilon_1. \end{aligned}$$

Now for $l = 1, 2$ we get

$$\begin{aligned} & \left\| c_l b c_l^* - \sum_{j=0}^{n-1} e_j \alpha(a) e_j \right\| \\ &\leq \sum_{j=0}^{n-1} \| [y_j \alpha^j(e_0 v_0 e_0) y_j^*] e_j a e_j [y_j \alpha^j(e_0 v_0 e_0) y_j^*]^* - e_j \alpha(a) e_j \| \\ &< 5n\varepsilon_1. \end{aligned}$$

Putting everything together, for $l = 1, 2$ we now get

$$\begin{aligned} & \|v_l(e a e)v_l^* - \alpha(e a e)\| \\ &\leq 2\|v_l - c_l\| + 2\|e a e - b\| + \|c_l b c_l^* - \alpha(b)\| \\ &\leq 2\|v_l - c_l\| + 2\|e a e - b\| \\ &\quad + \left\| c_l b c_l^* - \sum_{j=0}^{n-1} e_j \alpha(a) e_j \right\| + \left\| \alpha(b) - \sum_{j=0}^{n-1} e_j \alpha(a) e_j \right\| \\ &< \frac{2}{13}\varepsilon + 4n^2\varepsilon_1 + 5n\varepsilon_1 + 2n\varepsilon_1 < \varepsilon. \end{aligned}$$

We now have Condition (3) of Lemma 3.3, for both v_1 and v_2 .

The proof of Conditions (4) and (5) of Lemma 3.3 is just like in the proof of that lemma. We have

$$1 - e = 1 - f + f(1 - p) \lesssim g_1 + g_2 \in \overline{x A x},$$

which gives Condition (4). Moreover,

$$1 - e \lesssim g_1 + g_2 \leq 1 - \sum_{k=3}^{2N+2} g_k,$$

so $\sum_{k=3}^{2N+2} g_k \lesssim e$ by cancellation, and there are N mutually orthogonal projections dominated by $\sum_{k=3}^{2N+2} g_k$, each Murray-von Neumann equivalent to $1 - e$. \blacksquare

4. DUALITY

In this section we show that if an action of \mathbf{Z}_n on a simple unital C*-algebra has the tracial Rokhlin property and its generator is tracially approximately inner, then the dual action also has the tracial Rokhlin property. We need to impose some conditions on comparison of projections in order to make the proofs work. These will be automatically satisfied in the cases we are interested in, because in those cases both the original algebra and the crossed product satisfy the much stronger

condition that the order on K_0 is determined by traces. The result is related to Lemma 3.8 of [29].

At the end of this section, we use these results to prove a version of Theorem 2.7 for the fixed point algebra in place of the crossed product.

The following cancellation condition is needed in the main theorem of this section.

Definition 4.1. Let D be a unital C^* -algebra. We say that the *weak divisibility property* holds for projections in D if whenever $N \in \mathbf{N}$ and e_1, e_2, \dots, e_N and f_1, f_2, \dots, f_{N+1} are two sets of mutually Murray-von Neumann equivalent orthogonal projections in D such that $\sum_{k=1}^{N+1} f_k \precsim \sum_{k=1}^N e_k$, then $f_1 \precsim e_1$.

Lemma 4.2. Let A be a simple unital C^* -algebra with tracial rank zero. Then the projections in A have the weak divisibility property.

Proof. Without loss of generality $e_1 \neq 0$. For any tracial state τ on A , we have $\tau(e_1) > 0$ and $(N+1)\tau(f_1) \leq N\tau(e_1)$, so $\tau(f_1) < \tau(e_1)$. By Theorem 2.2, the order on projections over A is determined by traces, so $f_1 \precsim e_1$. ■

Theorem 4.3. Let A be a simple unital C^* -algebra and let $\alpha \in \text{Aut}(A)$ be tracially approximately inner and satisfy $\alpha^n = \text{id}_A$. Suppose that the action of \mathbf{Z}_n generated by α has the tracial Rokhlin property. Assume that A has cancellation of projections, and that the projections in $C^*(\mathbf{Z}_n, A, \alpha)$ have the weak divisibility property (Definition 4.1). Then the dual action of \mathbf{Z}_n on $C^*(\mathbf{Z}_n, A, \alpha)$ has the tracial Rokhlin property and its generator is tracially approximately inner.

Proof. Let $B = C^*(\mathbf{Z}_n, A, \alpha)$, and let $u \in B$ be the standard unitary (satisfying $uau^* = \alpha(a)$ for $a \in A$). Let $\widehat{\alpha} \in \text{Aut}(B)$ be the automorphism which generates the dual action. Thus, with $\omega = \exp(2\pi i/n)$, we have $\widehat{\alpha}(u) = \omega u$ and $\widehat{\alpha}(a) = a$ for $a \in A$.

When we verify the definition of the tracial Rokhlin property and of tracial approximate innerness for $\widehat{\alpha}$, we may clearly assume that the finite set has the form $F = F_0 \cup \{u\}$ for some α -invariant finite subset F_0 of the unit ball of A . We now claim that we may take the nonzero positive element $x \in B$ to in A . Theorem 4.2 of [31] shows that if A has Property (SP), then so does B . Moreover, B is simple by Corollary 1.12, so Theorem 4.2 of [31] also shows that if B has Property (SP), then so does $C^*(\mathbf{Z}_n, B, \widehat{\alpha}) \cong M_n(A)$. Property (SP) obviously passes to hereditary subalgebras, so we see that A has Property (SP) if and only if B has Property (SP). If A and B do not have Property (SP), then by Lemma 1.6 and Remark 3.2 we may as well take $x = 0$. If A and B have Property (SP), and if $x \in B$ is a nonzero positive element, then we can choose a nonzero projection $q_0 \in \overline{x B x}$, and use Theorem 4.2 of [31] to find a nonzero projection $p \in A$ such that $p \precsim q_0$ in B . It clearly suffices to use p in place of x . This proves the claim.

Accordingly, let F_0 be an α -invariant finite subset of the unit ball of A and set $F = F_0 \cup \{u\}$, let $\varepsilon > 0$, let $N \in \mathbf{N}$, and let $x \in A$ be a nonzero positive element. Without loss of generality $\varepsilon < 1$.

Choose a projection $e \in A$ and unitaries $v_1, v_2 \in eAe$ following Proposition 3.6, with F_0 in place of F , with $(n+2)^{-1}\varepsilon$ in place of ε , with $N(n+1)$ in place of N , and with x as given. In B , since

$$u^n = 1, \quad v_1^n = e, \quad ueu^* = \alpha(e) = e, \quad \text{and} \quad uv_1u^* = \alpha(v_1) = v_1,$$

we have $(u^*v_1)^n = e$. Moreover, for $a \in F_0$ we have

$$\|v_1(eae)v_1^* - \alpha(eae)\| < \frac{\varepsilon}{n+2} \quad \text{and} \quad u(eae)u^* = \alpha(eae),$$

so

$$\|(u^*v_1)(eae)(u^*v_1)^* - eae\| < \frac{\varepsilon}{n+2}.$$

Therefore, for $0 \leq j \leq n-1$ we have

$$\|(u^*v_1)^j(eae) - (eae)(u^*v_1)^j\| < j \left(\frac{\varepsilon}{n+2} \right) \leq \frac{n\varepsilon}{n+2}.$$

Define projections $e_j \in B$ by

$$e_j = \frac{1}{n} \sum_{k=0}^{n-1} (\omega^{-j} u^* v_1)^k$$

for $0 \leq j \leq n-1$. Then $\sum_{j=0}^{n-1} e_j = e$. Moreover,

$$\widehat{\alpha}(e_j) = \frac{1}{n} \sum_{k=0}^{n-1} (\omega^{-j} \overline{\omega} u^* v_1)^k = \frac{1}{n} \sum_{k=0}^{n-1} (\omega^{-(j+1)} u^* v_1)^k = e_{j+1}.$$

Also, u commutes with v_1 , hence with u^*v_1 , hence with all e_j , and for $a \in F_0$ we have

$$\begin{aligned} \|ae_j - e_j a\| &= \|aee_j - e_j ae\| \leq 2\|ea - ae\| + \|(eae)e_j - e_j(eae)\| \\ &\leq 2\|ea - ae\| + \frac{1}{n} \sum_{k=0}^{n-1} \|(eae)(u^*v_1)^k - (u^*v_1)^k(eae)\| \\ &< (2+n) \left(\frac{\varepsilon}{n+2} \right) = \varepsilon. \end{aligned}$$

We have verified Conditions (1) and (2) of Definition 1.1. For Condition (3), we observe that $1 - e$ is Murray-von Neumann equivalent to a projection in xAx . Condition (4) will be verified below.

We now set $v = u^*v_2$ and verify the first four conditions of Definition 3.1, using the same projection e as before. The first two and the fourth have already been done. For the third, first consider $a \in F_0$. We have

$$\|(u^*v_2)(eae)(u^*v_2)^* - eae\| < \frac{\varepsilon}{n+2}$$

for the same reason that this holds for v_1 in place of v_2 . So

$$\|v(eae)v^* - \widehat{\alpha}(eae)\| = \|v(eae)v^* - eae\| < \frac{\varepsilon}{n+2} \leq \varepsilon.$$

Furthermore,

$$vuv^* = u^*v_2uv_2^*u = \alpha^{-1}(v_2)v_2^*u = \omega v_2v_2^*u = \widehat{\alpha}(u).$$

It remains to verify the last condition of both definitions. We have

$$vv_1v^* = u^*v_2v_1v_2^*u = \alpha^{-1}(v_2v_1v_2^*) = (\omega v_2)v_1(\omega^{-1}v_2^*) = v_2v_1v_2^*.$$

Combining this with the computation of vuv^* and the estimate $\|v_2v_1v_2^* - v_1\| < (n+2)^{-1}\varepsilon$, we get

$$\|v(u^*v_1)v^* - \omega^{-1}u^*v_1\| < \frac{\varepsilon}{n+2}.$$

By induction,

$$\|v(u^*v_1)^k v^* - (\omega^{-1}u^*v_1)^k\| < \frac{k\varepsilon}{n+2}$$

for $k \geq 1$. Therefore

$$\begin{aligned} \|ve_jv^* - e_{j+1}\| &\leq \frac{1}{n} \sum_{k=0}^{n-1} \|v(\omega^{-j}u^*v_1)^k v^* - (\omega^{-(j+1)}u^*v_1)^k\| \\ &< \frac{1}{n} \sum_{k=0}^{n-1} \frac{k\varepsilon}{n+2} \leq \varepsilon. \end{aligned}$$

Since $\varepsilon < 1$, it follows that e_j is unitarily equivalent to e_{j+1} .

By construction there are $N(n+1)$ mutually orthogonal projections

$$g_1, g_2, \dots, g_{N(n+1)} \leq e,$$

each of which is Murray-von Neumann equivalent to $1 - e$. For $0 \leq m \leq n$ set

$$h_m = \sum_{k=1}^N g_{mN+k},$$

which are $n+1$ mutually orthogonal projections in eAe , each of which is Murray-von Neumann equivalent to h_0 . The weak divisibility property for projections in B implies that h_0 is Murray-von Neumann equivalent in B to a subprojection of e_0 . Thus there are N mutually orthogonal projections in e_0Be_0 , each of which is Murray-von Neumann equivalent to $1 - e$. Since e_0, e_1, \dots, e_{n-1} are all unitarily equivalent in B , the same is true for e_jBe_j for $0 \leq j \leq n-1$. This is Condition (4) of Definition 1.1. Since $e_0 \leq e$, we also have Condition (5) of Definition 3.1. ■

Corollary 4.4. Let A be a simple unital C*-algebra and let $\alpha \in \text{Aut}(A)$ be tracially approximately inner and satisfy $\alpha^n = \text{id}_A$. Suppose that the action of \mathbf{Z}_n generated by α has the tracial Rokhlin property. Assume that A has cancellation of projections. Then A has tracial rank zero if and only if $C^*(\mathbf{Z}_n, A, \alpha)$ has tracial rank zero.

Proof. If A has tracial rank zero, the conclusion is Theorem 2.7. So suppose that $C^*(\mathbf{Z}_n, A, \alpha)$ has tracial rank zero. Then the projections in $C^*(\mathbf{Z}_n, A, \alpha)$ have the weak divisibility property by Lemma 4.2. Theorem 4.3 applies, so that the dual action of \mathbf{Z}_n on $C^*(\mathbf{Z}_n, A, \alpha)$ has the tracial Rokhlin property. Therefore Theorem 2.7 implies that the crossed product by the dual action has tracial rank zero. Since this crossed product is isomorphic to $M_n \otimes A$, it follows from Theorem 3.12(1) of [37] that A has tracial rank zero. ■

Now we turn to the fixed point algebra.

Proposition 4.5. Let A be a C*-algebra, let G be a compact group, and let $\alpha: G \rightarrow \text{Aut}(A)$ be a continuous action of G on A . Suppose $C^*(G, A, \alpha)$ is simple. Then the fixed point algebra A^α is simple, is isomorphic to a full hereditary subalgebra of $C^*(G, A, \alpha)$, and is strongly Morita equivalent to $C^*(G, A, \alpha)$.

Proof. See the Proposition, Corollary, and proof of the Corollary in [56]. ■

Corollary 4.6. Let A be a simple unital C^* -algebra, and let $\alpha \in \text{Aut}(A)$ be an automorphism which satisfies $\alpha^n = \text{id}_A$ and such that the action of \mathbf{Z}_n generated by α has the tracial Rokhlin property. Suppose that A has tracial rank zero. Then the fixed point algebra A^α has tracial rank zero.

Proof. Theorem 2.7 implies that $C^*(\mathbf{Z}_n, A, \alpha)$ has tracial rank zero. This algebra is simple by Proposition 1.12. It therefore follows from Proposition 4.5 that A^α is isomorphic to a hereditary subalgebra in $C^*(\mathbf{Z}_n, A, \alpha)$. So Theorem 3.12(1) of [37] implies that A^α has tracial rank zero. ■

Corollary 4.7. Let A be a simple unital C^* -algebra, and let $\alpha \in \text{Aut}(A)$ be a tracially approximately inner automorphism which satisfies $\alpha^n = \text{id}_A$ and such that the action of \mathbf{Z}_n generated by α has the tracial Rokhlin property. Assume that A has cancellation of projections, and that the fixed point algebra A^α has tracial rank zero. Then A has tracial rank zero.

Proof. The crossed product $C^*(\mathbf{Z}_n, A, \alpha)$ is simple by Proposition 1.12. It therefore follows from Proposition 4.5 that A^α and $C^*(\mathbf{Z}_n, A, \alpha)$ are strongly Morita equivalent. Since everything is unital, $C^*(\mathbf{Z}_n, A, \alpha)$ is isomorphic to a hereditary subalgebra in some matrix algebra over A^α . Therefore Theorems 3.10 and 3.12(1) of [37] imply that $C^*(\mathbf{Z}_n, A, \alpha)$ has tracial rank zero. Now Corollary 4.4 shows that A has tracial rank zero. ■

5. HIGHER DIMENSIONAL NONCOMMUTATIVE TORUSES

In this section and the next two, we prove that every simple higher dimensional noncommutative torus is an AT algebra.

The first result in this direction was the Elliott-Evans Theorem [15] for the ordinary irrational rotation algebras, which we use here as the initial step of an induction argument. Without giving a complete list of later work, we mention four highlights. All simple three dimensional noncommutative toruses were shown to be AT algebras in [43]. In arbitrary dimension, “most” simple higher dimensional noncommutative toruses were shown to be AT algebras in [6]. Corollary 6.6 of [35] gives this result in all cases in which, in the skew symmetric matrix giving the commutation relations, the entries above the diagonal are rationally independent, as well as some others. Theorem 3.14 of [41] shows that the crossed product of $(S^1)^d$ by a minimal rotation is an AT algebra; in this case, most of the entries of the relevant skew symmetric matrix are zero.

The proof of Corollary 6.6 of [35] is an induction argument: if, when the non-commutative torus is written as a successive crossed product by actions of \mathbf{Z} , all the intermediate crossed products are simple, then the main results of [35] reduce the problem to the Elliott-Evans Theorem. One has some choice here: different choices of the commutation relations may well give the same C^* -algebra. As a very simple example, one might simply write the generators in a different order. Unfortunately, it seems not to be possible in general to choose commutation relations to give the same algebra, or even a Morita equivalent algebra (see [55]), and in such a way that the method of Corollary 6.6 of [35] applies, or even in such a way as to get a tensor product of algebras to which this method applies. However, if one allows one more kind of modification, namely the replacements of unitary generators by integer powers of themselves, then it is always possible to replace a noncommutative torus by a tensor product of algebras covered by Corollary 6.6 of [35]. The

new algebra isn't isomorphic, or even Morita equivalent, to the original. But if one replaces only one generator, the new algebra is the fixed point algebra of a tracially approximately inner action of a finite cyclic group which has the tracial Rokhlin property. This operation thus preserves tracial rank zero. Because K_0 and K_1 are torsion free, the classification theorem for simple nuclear C^* -algebras with tracial rank zero, Theorem 5.2 of [42], shows that this operation preserves the property of being an AT algebra.

In this section we present, in a form convenient for our purposes, some standard facts about higher dimensional noncommutative toruses. In Section 6, we prove that the relevant actions have the tracial Rokhlin property, and in Section 7 we show how to combine this result with [15], [35], and [42] to prove that all simple higher dimensional noncommutative toruses are AT algebras.

Notation 5.1. Let θ be a skew symmetric real $d \times d$ matrix. The *noncommutative torus* A_θ is by definition [54] the universal C^* -algebra generated by unitaries u_1, u_2, \dots, u_d subject to the relations

$$u_k u_j = \exp(2\pi i \theta_{j,k}) u_j u_k$$

for $1 \leq j, k \leq d$. (Of course, if all $\theta_{j,k}$ are integers, it is not really noncommutative.)

Some authors use $\theta_{k,j}$ in the commutation relation instead. See for example [34].

Remark 5.2. We note (see the beginning of Section 4 of [52] and the introduction to [55]) that A_θ is the universal C^* -algebra generated by unitaries u_x , for $x \in \mathbf{Z}^d$, subject to the relations

$$u_y u_x = \exp(\pi i \langle x, \theta(y) \rangle) u_{x+y}$$

for $x, y \in \mathbf{Z}^d$.

It follows that if $B \in \mathrm{GL}_d(\mathbf{Z})$, and if B^t denotes the transpose of B , then $A_{B^t \theta B} \cong A_\theta$. That is, A_θ is unchanged if θ is rewritten in terms of some other basis of \mathbf{Z}^d .

Remark 5.3. Let α be a skew symmetric real bicharacter on \mathbf{Z}^d , that is, a \mathbf{Z} -bilinear function $\alpha: \mathbf{Z}^d \times \mathbf{Z}^d \rightarrow \mathbf{R}$ such that $\alpha(x, y) = -\alpha(y, x)$ for all $x, y \in \mathbf{Z}^d$. For any basis (b_1, b_2, \dots, b_d) of \mathbf{Z}^d , there is a unique skew symmetric real $d \times d$ matrix θ such that

$$\alpha \left(\sum_{k=1}^d x_k b_k, \sum_{k=1}^d y_k b_k \right) = \sum_{j,k=1}^d x_j \theta_{j,k} y_k$$

for all $x, y \in \mathbf{Z}^d$. We define $A_\alpha = A_\theta$. Remark 5.2 shows that this C^* -algebra is independent of the choice of basis.

Remark 5.4. Let α be a skew symmetric real bicharacter on \mathbf{Z}^d , and let $H \subset \mathbf{Z}^d$ be a subgroup. Then $H \cong \mathbf{Z}^m$ for some $m \leq d$. By abuse of notation, we write $\alpha|_H$ for the restriction of α to $H \times H \subset \mathbf{Z}^d \times \mathbf{Z}^d$. There is a noncommutative torus $A_{\alpha|_H}$ by Remark 5.3, which does not depend on the choice of the isomorphism $H \cong \mathbf{Z}^m$.

For a skew symmetric real $d \times d$ matrix θ and a subgroup $H \subset \mathbf{Z}^d$ with a specified ordered basis, we write $\theta|_H$ for the matrix in that basis of the restriction to H of the real bicharacter $(x, y) \mapsto \langle x, \theta y \rangle$. For subgroups such as $\mathbf{Z}^m \times \{0\}$ or $\mathbf{Z}^m \times \{0\} \times \mathbf{Z}^l$, we use without comment the obvious basis.

We formalize a remark made in 1.7 of [13], according to which all noncommutative toruses can be obtained as successive crossed product by \mathbf{Z} .

Lemma 5.5. Let α be a skew symmetric real bicharacter on \mathbf{Z}^d . Then there is an automorphism φ of $A_{\alpha|_{\mathbf{Z}^{d-1} \times \{0\}}}$ which is homotopic to the identity and such that

$$A_\alpha \cong C^*(\mathbf{Z}, A_{\alpha|_{\mathbf{Z}^{d-1} \times \{0\}}}, \varphi).$$

Proof. Let θ be the matrix of α in the standard basis. Let $\beta = \alpha|_{\mathbf{Z}^{d-1} \times \{0\}}$. Then the matrix of β is $(\theta_{j,k})_{1 \leq j, k \leq d-1}$. Let u_1, u_2, \dots, u_{d-1} be the standard generators of A_β . Then φ is determined by $\varphi(u_j) = \exp(2\pi i \alpha_{j,d}) u_j$. It is clear that φ is homotopic to the identity. ■

The following definition is essentially from Section 1.1 of [58].

Definition 5.6. The skew symmetric real $d \times d$ matrix θ is *nondegenerate* if whenever $x \in \mathbf{Z}^d$ satisfies $\exp(2\pi i \langle x, \theta y \rangle) = 1$ for all $y \in \mathbf{Z}^d$, then $x = 0$. Otherwise, θ is *degenerate*. We similarly refer to degeneracy and nondegeneracy of a skew symmetric real bicharacter on \mathbf{Z}^d .

Lemma 5.7. Let θ be a skew symmetric real $d \times d$ matrix. Then θ is degenerate if and only if there exists $x \in \mathbf{Q}^d \setminus \{0\}$ such that $\langle x, \theta y \rangle \in \mathbf{Q}$ for all $y \in \mathbf{Q}^d$.

Proof. If θ is degenerate, choose $w \neq 0$ such that $\exp(2\pi i \langle w, \theta y \rangle) = 1$ for all $y \in \mathbf{Z}^d$. Then $\langle w, \theta y \rangle \in \mathbf{Z}$ for all $y \in \mathbf{Z}^d$. If now $y \in \mathbf{Q}^d$ is arbitrary, then there exists $m \in \mathbf{Z} \setminus \{0\}$ such that $my \in \mathbf{Z}^d$. So

$$\langle w, \theta y \rangle = \frac{1}{m} \langle w, \theta(my) \rangle \in \frac{1}{m} \mathbf{Z} \subset \mathbf{Q}.$$

Conversely, assume $x \in \mathbf{Q}^d \setminus \{0\}$ and $\langle x, \theta y \rangle \in \mathbf{Q}$ for all $y \in \mathbf{Q}^d$. Choose $m \in \mathbf{Z}$ with $m > 0$ such that $m \langle x, \theta \delta_k \rangle \in \mathbf{Z}$ for $1 \leq k \leq d$. Then $mx \neq 0$ and $\exp(2\pi i \langle mx, \theta y \rangle) = 1$ for all $y \in \mathbf{Z}^d$. ■

Lemma 5.8. Let θ be a skew symmetric real $d \times d$ matrix. Let $B \in \mathrm{GL}_d(\mathbf{Q})$. Then $B^\dagger \theta B$ is nondegenerate if and only if θ is nondegenerate.

Proof. It suffices to prove one direction. Suppose θ is degenerate. By Lemma 5.7, there is $x \in \mathbf{Q}^d \setminus \{0\}$ such that $\langle x, \theta y \rangle \in \mathbf{Q}$ for all $y \in \mathbf{Q}^d$. Then $B^{-1}x \in \mathbf{Q}^d \setminus \{0\}$ and

$$\langle B^{-1}x, B^\dagger \theta B y \rangle = \langle x, \theta B y \rangle \in \mathbf{Q}$$

for all $y \in \mathbf{Q}^d$. So $B^\dagger \theta B$ is degenerate. ■

The following result is well known.

Theorem 5.9. The C^* -algebra A_θ of Notation 5.1 is simple if and only if θ is nondegenerate. Moreover, if A_θ is simple it has a unique tracial state.

Proof. If θ is nondegenerate, then A_θ is simple by Theorem 3.7 of [58]. (Note the standing assumption of nondegeneracy throughout Section 3 of [58].)

When A_θ is simple, the proof of Lemma 3.1 of [58] shows that A_θ can have at most one tracial state. Existence of a tracial state is well known, or can be obtained from Lemma 5.5 by induction on n .

If θ is degenerate, then we follow 1.8 of [13]. Choose $n \in \mathbf{Z}^d \setminus \{0\}$ such that $\exp(2\pi i \langle n, \theta y \rangle) = 1$ for all $y \in \mathbf{Z}^d$. Then $v = u_1^{n_1} u_2^{n_2} \cdots u_d^{n_d}$ is a nontrivial element of the center of A_θ , which is therefore not simple. ■

6. THE TRACIAL ROKHLIN PROPERTY AND HIGHER DIMENSIONAL NONCOMMUTATIVE TORUSES

In this section, we prove that if θ is nondegenerate, then the action of \mathbf{Z}_n which multiplies one of the standard generators of A_θ by a primitive n -th root of 1 has the tracial Rokhlin property. We note for comparison the related result in Section 6 of [34], that if $\alpha \in \text{Aut}(A_\theta)$ is of the form $\alpha(u_j) = \lambda_j u_j$, with $\lambda_1, \lambda_2, \dots, \lambda_n \in S^1$, and if all positive powers of α are outer, then α has the Rokhlin property.

As is done in the proof in [4] that A_θ has real rank zero, and analogously to Section 6 of [34], we will reduce to a construction in the ordinary irrational rotation algebras. We therefore begin with several facts about these algebras.

For reference, and to establish notation, we state the following theorem. Its proof is contained in Theorem 1.1 and Proposition 1.3 of [1]. Also see Corollary 3.6 and Definition 3.3 of [53]. We refer to [11] for information on continuous fields of C^* -algebras. See especially Sections 10.1 and 10.3. We use v and w for the generators of the irrational rotation algebra, to avoid confusion with the generators u_1, u_2, \dots, u_d of a higher dimensional noncommutative torus A_θ .

Theorem 6.1. For $\eta \in \mathbf{R}$ let A_η be the rotation algebra, the universal C^* -algebra generated by unitaries v_η and w_η satisfying $w_\eta v_\eta = \exp(2\pi i \eta) v_\eta w_\eta$. Let A be the C^* -algebra of the discrete Heisenberg group, which is the universal C^* -algebra generated by unitaries v, w, z subject to the relations

$$wv = zvw, \quad zv = vz, \quad \text{and} \quad zw = wz.$$

Then there is a continuous field of C^* -algebras over S^1 whose fiber over $\exp(2\pi i \eta)$ is A_η , whose C^* -algebra of continuous sections is A , and such that the evaluation map $\text{ev}_\eta: A \rightarrow A_\eta$ of sections at $\exp(2\pi i \eta)$ is determined by

$$\text{ev}_\eta(v) = v_\eta, \quad \text{ev}_\eta(w) = w_\eta, \quad \text{and} \quad \text{ev}_\eta(z) = \exp(2\pi i \eta) \cdot 1.$$

Since we will only formally deal with one continuous field in this section, the following notation is unambiguous.

Notation 6.2. For a subset $E \subset S^1$, we let $\Gamma(E)$ be the set of continuous sections of the continuous field of Theorem 6.1 over E . (See 10.1.6 of [11].)

For any such section a , we further write $a(\eta)$ for $a(\exp(2\pi i \eta))$. No confusion should arise.

Lemma 6.3. Let the notation be as in Theorem 6.1 and Notation 6.2. Let τ_η be the standard trace on A_η , satisfying $\tau(1) = 1$ and $\tau_\eta(v_\eta^m w_\eta^n) = 0$ unless $m = n = 0$. Let $U \subset S^1$ be an open set, and let $a \in \Gamma(U)$. Then $\eta \mapsto \tau_\eta(a(\eta))$ is continuous.

Proof. We check continuity at η_0 . Choose a continuous function $h: S^1 \rightarrow [0, 1]$ such that $\text{supp}(h) \subset U$ and such that $h = 1$ on a neighborhood of η_0 . Then it suffices to consider the continuous section ha in place of a . Now ha is the restriction to U of a continuous section b defined on all of S^1 , satisfying $b(\zeta) = 0$ for $\zeta \notin U$. Accordingly, we may restrict to the case $U = S^1$. Then $a \in A$.

From the formulas

$$\text{ev}_\eta(v) = v_\eta, \quad \text{ev}_\eta(w) = w_\eta, \quad \text{and} \quad \text{ev}_\eta(z) = \exp(2\pi i \eta) \cdot 1$$

and the definition of τ_η , it is immediate that if b is any (noncommutative) monomial in v, w, z , and their adjoints, then $\eta \mapsto \tau_\eta(b(\eta))$ is continuous. Therefore the

same holds for any noncommutative polynomial, and hence for any norm limit of noncommutative polynomials, including a . ■

Lemma 6.4. Let the notation be as in Theorem 6.1 and Lemma 6.3. Let $\eta \in \mathbf{R} \setminus \mathbf{Q}$. Let $n \in \mathbf{N}$, let $\omega = \exp(2\pi i/n)$, and let $\alpha: A_\eta \rightarrow A_\eta$ be the unique automorphism satisfying $\alpha(v_\eta) = \omega v_\eta$ and $\alpha(w_\eta) = w_\eta$. Then for every $\varepsilon > 0$ there exist mutually orthogonal projections e_0, e_1, \dots, e_{n-1} such that (with $e_n = e_0$) we have $\alpha(e_j) = e_{j+1}$ for $0 \leq j \leq n-1$, and such that $1 - n\tau_\eta(e_0) < \varepsilon$.

Proof. Set $\varepsilon_0 = \frac{1}{4n}\varepsilon$. Let $f: S^1 \rightarrow [0, 1]$ be a continuous function such that $\text{supp}(f)$ is contained in the open arc from 1 to ω , and such that $f(\zeta) = 1$ for all ζ in the closed arc from $\exp(2\pi i\varepsilon_0)$ to $\exp(2\pi i[\frac{1}{n} - \varepsilon_0])$. Then $f(v_\eta)$ is a positive element of A_η with $\|f(v_\eta)\| \leq 1$ and $\tau_\eta(f(v_\eta)) \geq \frac{1}{n} - 2\varepsilon_0$. Since A_η has real rank zero (see Remark 6 of [15], or Theorem 1.5 of [5]), there is a projection e_0 in the hereditary subalgebra B of A_η generated by $f(v_\eta)$ such that $\|e_0 f(v_\eta) - f(v_\eta)\| < \varepsilon_0$. Therefore $\|e_0 f(v_\eta) e_0 - f(v_\eta)\| < 2\varepsilon_0$. Since $e_0 f(v_\eta) e_0 \leq e_0$, it follows that

$$\tau_\eta(e_0) \geq \tau_\eta(e_0 f(v_\eta) e_0) > \tau_\eta(f(v_\eta)) - 2\varepsilon_0 \geq \frac{1}{n} - 4\varepsilon_0.$$

We have $\alpha^k(f(v_\eta))\alpha^l(f(v_\eta)) = 0$ for $0 \leq k, l \leq n-1$ and $k \neq l$. Therefore $\alpha^k(B)\alpha^l(B) = \{0\}$ for such k and l , whence also $\alpha^k(e_0)\alpha^l(e_0) = 0$. Define $e_k = \alpha^k(e_0)$ for $0 \leq k \leq n-1$. Then e_0, e_1, \dots, e_{n-1} are mutually orthogonal projections such that $\alpha(e_j) = e_{j+1}$ for $0 \leq j \leq n-1$. Moreover,

$$1 - n\tau_\eta(e_0) < 1 - n\left(\frac{1}{n} - 4\varepsilon_0\right) = 4n\varepsilon_0 = \varepsilon,$$

as desired. ■

We now return to the higher dimensional noncommutative toruses. The idea is to find an approximately central copy of an ordinary irrational rotation algebra A_η , such that the restriction to it of our action is the one in Lemma 6.4. Since the projections in A_η must be chosen ahead of time, at least approximately, we must require that η be arbitrarily close to some fixed η_0 . Nondegeneracy enters through Lemma 6.7 below. To obtain the correct restricted action, we use the condition (3) in Lemma 6.12 below. From then on, we roughly follow the argument used in [5] to prove approximate divisibility. We vary the arrangement slightly to make part of the argument easily available for later use.

Definition 6.5. Let θ be a skew symmetric real $d \times d$ matrix. Let

$$n = (n_1, n_2, \dots, n_d) \in \mathbf{Z}^d \quad \text{and} \quad v = u_1^{n_1} u_2^{n_2} \cdots u_d^{n_d} \in A_\theta.$$

We write γ_n for the inner automorphism $\text{Ad}(v)$ of the noncommutative torus A_θ . We further define a homomorphism $\sigma: \mathbf{Z}^d \rightarrow (S^1)^d$ by the formula $\sigma(n)_j = \exp(2\pi i(\theta n)_j)$ for $1 \leq j \leq d$. (Here the expression θn is the usual action of a $d \times d$ matrix on an element of \mathbf{R}^d .)

Lemma 6.6. Let θ be a skew symmetric real $d \times d$ matrix. With γ and σ as in Definition 6.5, we have $\gamma_n(u_j) = \sigma(n)_j u_j$ for $n \in \mathbf{Z}^d$ and $1 \leq j \leq d$. Moreover, if $m \in \mathbf{Z}^d$, then

$$\gamma_n(u_1^{m_1} u_2^{m_2} \cdots u_d^{m_d}) = \exp(2\pi i\langle m, \theta n \rangle) u_1^{m_1} u_2^{m_2} \cdots u_d^{m_d}$$

for all $n \in \mathbf{Z}^d$.

Proof. The first formula is the special case of the second obtained by setting $m = \delta_j$, the j -th standard basis vector of \mathbf{Z}^d . By linearity, both formulas will follow if we check the first when $m = \delta_j$ and $n = \delta_k$. Since $(\theta\delta_k)_j = \theta_{j,k}$, this is just the commutation relation

$$u_k u_j u_k^* = \exp(2\pi i \theta_{j,k}) u_j,$$

which is the same as the one in from Notation 5.1. ■

Lemma 6.7. Let θ be a skew symmetric real $d \times d$ matrix. The homomorphism $\sigma: \mathbf{Z}^d \rightarrow (S^1)^d$ of Definition 6.5 has dense range if and only if θ is nondegenerate.

Proof. Assume σ does not have dense range. Let $H = \overline{\sigma(\mathbf{Z}^d)}$, which is a proper closed subgroup of $(S^1)^d$. Choose a nontrivial character $\mu: (S^1)^d \rightarrow S^1$ whose kernel contains H . By the identification of the dual group of $(S^1)^d$, there is $r \in \mathbf{Z}^d \setminus \{0\}$ such that

$$\mu(\zeta_1, \zeta_2, \dots, \zeta_d) = \zeta_1^{r_1} \zeta_2^{r_2} \cdots \zeta_d^{r_d}$$

for all $\zeta \in (S^1)^d$. Because $H \subset \text{Ker}(\mu)$, for all $n \in \mathbf{Z}^d$ we have

$$\begin{aligned} 1 &= \mu(\sigma(n)) = \exp(2\pi i (\theta n)_1)^{r_1} \exp(2\pi i (\theta n)_2)^{r_2} \cdots \exp(2\pi i (\theta n)_d)^{r_d} \\ &= \exp(2\pi i \langle r, \theta n \rangle). \end{aligned}$$

Thus θ is degenerate.

Now suppose that θ is degenerate. Then we may choose $r \in \mathbf{Z}^d \setminus \{0\}$ such that $\exp(2\pi i \langle r, \theta n \rangle) = 1$ for all $n \in \mathbf{Z}^d$. Reversing the above calculation, we find that the nontrivial character

$$\mu(\zeta_1, \zeta_2, \dots, \zeta_d) = \zeta_1^{r_1} \zeta_2^{r_2} \cdots \zeta_d^{r_d}$$

satisfies $\mu(\sigma(n)) = 1$ for all $n \in \mathbf{Z}^d$. Therefore σ does not have dense range. ■

Corollary 6.8. Let θ be a nondegenerate skew symmetric real $d \times d$ matrix. Let $G \subset \mathbf{Z}^d$ be a subgroup with finite index. Let $\sigma: \mathbf{Z}^d \rightarrow (S^1)^d$ be the homomorphism of Definition 6.5. Then $\sigma(G)$ is dense in $(S^1)^d$.

Proof. Let $H = \overline{\sigma(G)}$. Let S be a set of coset representatives for G in \mathbf{Z}^d . Then the sets $\sigma(m)H$, for $m \in S$, are closed and are pairwise equal or disjoint. By Lemma 6.7, their union is $(S^1)^d$. Since there are finitely many of them, and since $(S^1)^d$ is connected, it follows that all are equal to $(S^1)^d$. ■

Corollary 6.9. Let θ be a nondegenerate skew symmetric real $d \times d$ matrix. Let $\zeta_1, \zeta_2, \dots, \zeta_d \in S^1$. Let $\alpha \in A_\theta$ be the automorphism determined by $\alpha(u_j) = \zeta_j u_j$ for $1 \leq j \leq d$. Then α is approximately inner.

Proof. It suffices to find, for all $\varepsilon > 0$, a unitary $v \in A_\theta$ such that $\|\alpha(u_j) - vu_j v^*\| < \varepsilon$ for $1 \leq j \leq d$. Choose $\delta > 0$ small enough that if $(\omega_1, \omega_2, \dots, \omega_d) \in (S^1)^d$ satisfies

$$d((\omega_1, \omega_2, \dots, \omega_d), (\zeta_1, \zeta_2, \dots, \zeta_d)) < \delta,$$

then $|\omega_j - \zeta_j| < \varepsilon$ for $1 \leq j \leq d$. Then use Lemma 6.7 to choose $n \in \mathbf{Z}^d$ such that $d(\sigma(n), (\zeta_1, \zeta_2, \dots, \zeta_d)) < \delta$. Take $v = u_1^{n_1} u_2^{n_2} \cdots u_d^{n_d}$ and use Lemma 6.6. ■

The following corollary seems to be of interest, but will not be used here.

Corollary 6.10. Let θ be a nondegenerate skew symmetric real $d \times d$ matrix. Let $n \in \mathbf{N}$ and let $\omega = \exp(2\pi i/n)$. Let $1 \leq k \leq d$, and let $\alpha \in A_\theta$ be the automorphism determined by $\alpha(u_k) = \omega u_k$ and $\alpha(u_j) = u_j$ for $j \neq k$. Then for every finite set $F \subset A_\theta$ and every $\varepsilon > 0$ there is a unitary $v \in A_\theta$ such that $\alpha(v) = v$ and $\|\alpha(a) - vav^*\| < \varepsilon$ for $a \in F$.

Proof. Without loss of generality $k = 1$. The proof is the same as for Corollary 6.9, but using the finite index subgroup $n\mathbf{Z} \oplus \mathbf{Z}^{d-1}$ in place of \mathbf{Z}^d . This substitution is valid by Corollary 6.8. ■

Lemma 6.11. Let θ be a nondegenerate skew symmetric real $d \times d$ matrix. Let $n, N \in \mathbf{N}$, and let $1 \leq k \leq d$. Then for every $\varepsilon > 0$ there exists $l = (l_1, l_2, \dots, l_d) \in \mathbf{Z}^d$ such that:

- (1) $v = u_1^{l_1} u_2^{l_2} \cdots u_d^{l_d}$ satisfies $\|vu_j - u_jv\| < \varepsilon$ for $1 \leq j \leq d$.
- (2) $l_k = 1 \pmod{n}$.
- (3) There is j such that $|l_j| > N$.

Proof. Without loss of generality $k = 1$. Set $\alpha = \text{Ad}(u_1^*)$. There are $\zeta_1, \zeta_2, \dots, \zeta_d \in S^1$ such that $\alpha(u_j) = \zeta_j u_j$ for $1 \leq j \leq d$. Let $G = n\mathbf{Z} \oplus \mathbf{Z}^{d-1}$, which is a finite index subgroup of \mathbf{Z}^d . According to Corollary 6.8, the subgroup $\sigma(G)$ is dense in $(S^1)^d$. Let

$$F = \{l \in \mathbf{Z}^d : |l_j| \leq N + 1 \text{ for } 1 \leq j \leq d\}.$$

Since F is finite, $\sigma(G \setminus F)$ is also dense in $(S^1)^d$. Choose $\delta > 0$ small enough that if $(\omega_1, \omega_2, \dots, \omega_d) \in (S^1)^d$ satisfies

$$d((\omega_1, \omega_2, \dots, \omega_d), (\zeta_1, \zeta_2, \dots, \zeta_d)) < \delta,$$

then $|\omega_j - \zeta_j| < \varepsilon$ for $1 \leq j \leq d$. Then use density of $\sigma(G \setminus F)$ to choose $r \in G \setminus F$ such that $d(\sigma(r), (\zeta_1, \zeta_2, \dots, \zeta_d)) < \delta$. So with $v_0 = u_1^{r_1} u_2^{r_2} \cdots u_d^{r_d}$, we get $\|v_0 u_j v_0^* - u_1^* u_j u_1\| < \varepsilon$ for $1 \leq j \leq d$. Define

$$l = (r_1 + 1, r_2, \dots, r_d) \in \mathbf{Z}^d \quad \text{and} \quad v = u_1^{l_1} u_2^{l_2} \cdots u_d^{l_d} = u_1 v_0 \in A_\theta.$$

Clearly $\|vu_j v^* - u_j\| < \varepsilon$ for $1 \leq j \leq d$. We have $l_1 = 1 \pmod{n}$ because $r_1 \in n\mathbf{Z}$. We have $|l_j| > N$ for some j , because $|r_j| > N + 1$ for some j . ■

The next lemma is the analog in our context of Lemma 4.6 of [5].

Lemma 6.12. Let θ be a nondegenerate skew symmetric real $d \times d$ matrix. Let $n \in \mathbf{N}$, let $1 \leq k \leq d$, and let $\eta_0 \in \mathbf{R} \setminus \mathbf{Q}$. Then for every $\varepsilon > 0$ there exist

$$l = (l_1, l_2, \dots, l_d) \in \mathbf{Z}^d \quad \text{and} \quad m = (m_1, m_2, \dots, m_d) \in \mathbf{Z}^d$$

such that:

- (1) $v = u_1^{l_1} u_2^{l_2} \cdots u_d^{l_d}$ and $w = u_1^{m_1} u_2^{m_2} \cdots u_d^{m_d}$ satisfy $\|vu_j - u_jv\| < \varepsilon$ and $\|wu_j - u_jw\| < \varepsilon$ for $1 \leq j \leq d$.
- (2) There is $\eta \in \mathbf{R} \setminus \mathbf{Q}$ such that $|\exp(2\pi i\eta) - \exp(2\pi i\eta_0)| < \varepsilon$ and the unitaries v and w of Part (1) satisfy $wv = \exp(2\pi i\eta)vw$.
- (3) $l_k = 1 \pmod{n}$ and $m_k = 0 \pmod{n}$.

Proof. Without loss of generality $k = 1$ and $\eta_0 \in [-\frac{1}{2}, \frac{1}{2}]$. Choose $N \in \mathbf{N}$ so large that $2\pi/N < \varepsilon$. Use Lemma 6.11 with θ , n , and ε as given, with $k = 1$, and with this value of N , obtaining

$$l \in \mathbf{Z}^d \quad \text{and} \quad v = u_1^{l_1} u_2^{l_2} \cdots u_d^{l_d}.$$

Note in particular that $\|vu_jv^* - u_j\| < \varepsilon$ for $1 \leq j \leq d$ and $l_1 = 1 \pmod{n}$. Let s be an index such that $|l_s| > N$.

Let

$$T = \{\eta \in \mathbf{R} : (u_1^{r_1}u_2^{r_2} \cdots u_d^{r_d})v(u_1^{r_1}u_2^{r_2} \cdots u_d^{r_d})^* = \exp(2\pi i\eta)v \text{ for some } r \in \mathbf{Z}^d\}.$$

Then T is a subgroup of \mathbf{R} which is generated by $d+1$ elements, namely 1 and elements corresponding to letting r run through the standard basis vectors of \mathbf{Z}^d . So $T \cap \mathbf{Q}$ is also finitely generated, and is therefore discrete. Since $\eta_0 \notin \mathbf{Q}$, we have $\text{dist}(\eta_0, T \cap \mathbf{Q}) > 0$. Set $\varepsilon_0 = \min(\varepsilon, \text{dist}(\eta_0, T \cap \mathbf{Q}))$.

Set

$$M = \sum_{j=1}^d |l_j| \quad \text{and} \quad \delta = \min\left(\frac{1}{2}\varepsilon_0, M^{-1}\varepsilon_0\right).$$

Let G be the finite index subgroup $G = n\mathbf{Z} \oplus \mathbf{Z}^{d-1} \subset \mathbf{Z}^d$. Let

$$\lambda = (1, \dots, 1, \exp(2\pi i\eta_0/l_s), 1, \dots, 1) \in (S^1)^d,$$

where $\exp(2\pi i\eta_0/l_s)$ is in position s . Use Corollary 6.8 and Lemma 6.6 to choose $m \in G$ such that $\sigma(m)$, as in Definition 6.5, is so close to λ that $w = u_1^{m_1}u_2^{m_2} \cdots u_d^{m_d}$ satisfies $\|wu_jw^* - u_j\| < \delta$ for $j \neq s$, and $\|wu_sw^* - \exp(2\pi i\eta_0/l_s)u_s\| < \delta$.

Since $\delta \leq \varepsilon$, it is clear that $\|wu_jw^* - u_j\| < \varepsilon$ for $j \neq s$. Also

$$\|wu_sw^* - u_s\| \leq \|wu_sw^* - \exp(2\pi i\eta_0/l_s)u_s\| + |\exp(2\pi i\eta_0/l_s) - 1|.$$

Using $\delta \leq \frac{1}{2}\varepsilon$, the first term is less than $\frac{1}{2}\varepsilon$. The second term satisfies

$$|\exp(2\pi i\eta_0/l_s) - 1| < 2\pi \left| \frac{\eta_0}{l_s} \right| < 2\pi \left(\frac{1}{2N} \right) \leq \frac{1}{2}\varepsilon.$$

Therefore $\|wu_jw^* - u_j\| < \varepsilon$ for $j = s$ as well. This completes the verification of Part (1) of the conclusion. Part (3) holds because $m_1 \in n\mathbf{Z}$ by construction.

It remains to prove Part (2). For each j with $1 \leq j \leq d$, there is $\zeta_j \in S^1$ such that $wu_jw^* = \zeta_j u_j$. Then

$$wvw^* = \zeta_1^{l_1} \zeta_2^{l_2} \cdots \zeta_d^{l_d} v.$$

Thus $wv = \exp(2\pi i\eta)vw$ for some $\eta \in \mathbf{R}$. By construction we have $|\zeta_j - 1| < M^{-1}\varepsilon_0$ for $j \neq s$, and $|\zeta_s - \exp(2\pi i\eta_0/l_s)| < M^{-1}\varepsilon_0$. It follows that

$$\begin{aligned} |\zeta_1^{l_1} \zeta_2^{l_2} \cdots \zeta_d^{l_d} - \exp(2\pi i\eta_0/l_s)l_s| &\leq |l_s| \cdot |\zeta_s - \exp(2\pi i\eta_0/l_s)| + \sum_{j \neq s} |l_j| \cdot |\zeta_j - 1| \\ &< \sum_{j=1}^d |l_j| M^{-1}\varepsilon_0 \leq \varepsilon_0. \end{aligned}$$

Therefore

$$\|wv - \exp(2\pi i\eta_0)vw\| = |\zeta_1^{l_1} \zeta_2^{l_2} \cdots \zeta_d^{l_d} - \exp(2\pi i\eta_0)| < \varepsilon_0,$$

which is the same as $|\exp(2\pi i\eta) - \exp(2\pi i\eta_0)| < \varepsilon_0$. In particular, $|\exp(2\pi i\eta) - \exp(2\pi i\eta_0)| < \varepsilon$, as desired. Moreover, $\eta \in T$ and there is no $\rho \in T \cap \mathbf{Q}$ such that $|\exp(2\pi i\rho) - \exp(2\pi i\eta_0)| < \varepsilon_0$, whence $\eta \notin \mathbf{Q}$. \blacksquare

The proofs of the next two results together parallel the proof of Theorem 1.5 of [5]. The first of them says, roughly, that higher dimensional noncommutative

toruses contain approximately central copies of irrational rotation algebras, constructed in a special way. It will be used again later. Unfortunately, the rotation parameter varies with the degree of approximation.

Lemma 6.13. Let θ be a nondegenerate skew symmetric real $d \times d$ matrix, let $n \in \mathbf{N}$, and let $1 \leq k \leq d$. Then for every $\eta_0 \in \mathbf{R}$, every open set $U \subset S^1$ containing $\exp(2\pi i\eta_0)$, every finite subset $F \subset A_\theta$, every finite subset $S \subset \Gamma(U)$ (following Notation 6.2), and every $\varepsilon > 0$, there exist $\eta \in \mathbf{R} \setminus \mathbf{Q}$ and

$$l = (l_1, l_2, \dots, l_d) \in \mathbf{Z}^d \quad \text{and} \quad m = (m_1, m_2, \dots, m_d) \in \mathbf{Z}^d$$

such that:

- (1) $|\eta - \eta_0| < \varepsilon$ and $\exp(2\pi i\eta) \in U$.
- (2) $x = u_1^{l_1} u_2^{l_2} \cdots u_d^{l_d}$ and $y = u_1^{m_1} u_2^{m_2} \cdots u_d^{m_d}$ satisfy $yx = \exp(2\pi i\eta)xy$.
- (3) Following the notation of Theorem 6.1, and with x and y as in Part (2), let $\varphi: A_\eta \rightarrow A_\theta$ be the homomorphism such that $\varphi(v_\eta) = x$ and $\varphi(w_\eta) = y$. Then $\|[a, \varphi(b(\eta))]\| < \varepsilon$ for all $a \in F$ and all $b \in S$.
- (4) $l_k \equiv 1 \pmod{n}$ and $m_k \equiv 0 \pmod{n}$.

Proof. Let the notation be as in Theorem 6.1 and Notation 6.2.

Without loss of generality $\varepsilon < 1$. Then there is $\varepsilon_0 > 0$ such that whenever $\zeta \in S^1$ satisfies $|\zeta - \exp(2\pi i\eta_0)| < \varepsilon_0$, there is a unique $\eta \in \mathbf{R}$ such that $\exp(2\pi i\eta) = \zeta$ and $|\eta - \eta_0| < \varepsilon$.

Without loss of generality $\|a\| \leq 1$ for all $a \in F$. Replacing U by an open set V with $\exp(2\pi i\eta_0) \in V \subset \overline{V} \subset U$, we may assume every $b \in S$ is bounded. Then without loss of generality $\|b(\eta)\| \leq 1$ for all $b \in S$ and $\eta \in U$. Write $F = \{a_1, a_2, \dots, a_s\}$ and $S = \{b_1, b_2, \dots, b_t\}$. Choose polynomials g_1, g_2, \dots, g_t in four noncommuting variables such that

$$\|g_r(v_{\eta_0}, v_{\eta_0}^*, w_{\eta_0}, w_{\eta_0}^*) - b_r(\eta_0)\| < \frac{1}{7}\varepsilon$$

for $1 \leq r \leq t$. Because the rotation algebras form a continuous field over S^1 (Theorem 6.1), there is $\delta > 0$ such that whenever $|\eta - \eta_0| < \delta$ we have $\exp(2\pi i\eta_0) \in U$, and

$$\|g_r(v_\eta, v_\eta^*, w_\eta, w_\eta^*) - b_r(\eta)\| < \frac{2}{7}\varepsilon$$

for $1 \leq r \leq t$.

Choose polynomials f_1, f_2, \dots, f_t in $2d$ noncommuting variables such that

$$\|f_r(u_1, u_1^*, \dots, u_d, u_d^*) - a_r\| < \frac{\varepsilon}{7(1+\varepsilon)}$$

for $1 \leq r \leq s$. Choose (see Proposition 4.3 of [5]) $\delta_0 > 0$ such that whenever D is a C*-algebra and

$$c_1, c_2, \dots, c_{2d}, d_1, d_2, d_3, d_4 \in D$$

are elements of norm 1 which satisfy $\|[c_r, d_j]\| < \delta_0$ for all j and r , then

$$\|[f_r(c_1, c_2, \dots, c_{2d}), g_j(d_1, d_2, d_3, d_4)]\| < \frac{1}{7}\varepsilon$$

for $1 \leq r \leq s$ and $1 \leq j \leq t$.

Apply Lemma 6.12 with θ, n, η_0 , and k as given, and with $\min(\varepsilon_0, \delta, \delta_0)$ in place of ε . We obtain $\eta \in \mathbf{R} \setminus \mathbf{Q}$ and

$$l = (l_1, l_2, \dots, l_d) \in \mathbf{Z}^d \quad \text{and} \quad m = (m_1, m_2, \dots, m_d) \in \mathbf{Z}^d.$$

Set

$$x = u_1^{l_1} u_2^{l_2} \cdots u_d^{l_d} \quad \text{and} \quad y = u_1^{m_1} u_2^{m_2} \cdots u_d^{m_d}.$$

By the choice of ε_0 , we may assume that $|\eta - \eta_0| < \varepsilon$, and by the choice of δ we have $\exp(2\pi i \eta_0) \in U$. This is Part (1) of the conclusion. Parts (2) and (4) are immediate.

It remains to prove Part (3). Part (1) of the conclusion of Lemma 6.12 and the choice of δ_0 ensure that

$$\|[f_r(u_1, u_1^*, \dots, u_d, u_d^*), g_j(x, x^*, y, y^*)]\| < \frac{1}{7}\varepsilon$$

for $1 \leq r \leq s$ and $1 \leq j \leq t$. From the choice of δ , we get

$$\|g_j(x, x^*, y, y^*)\| < \|\varphi(b_j(\eta))\| + \frac{2}{7}\varepsilon < 1 + \varepsilon$$

for $1 \leq j \leq t$. Using the choice of the polynomials f_r , we therefore get

$$\begin{aligned} \| [a_r, \varphi(b_j(\eta))] \| &\leq 2\|a_r\| \cdot \|\varphi(b_j(\eta)) - g_j(x, x^*, y, y^*)\| \\ &\quad + 2\|a_r - f_r(u_1, u_1^*, \dots, u_d, u_d^*)\| \cdot \|g_j(x, x^*, y, y^*)\| \\ &\quad + \|[f_r(u_1, u_1^*, \dots, u_d, u_d^*), g_j(x, x^*, y, y^*)]\| \\ &< 2\left(\frac{2\varepsilon}{7}\right) + 2(1 + \varepsilon)\left(\frac{\varepsilon}{7(1 + \varepsilon)}\right) + \frac{\varepsilon}{7} = \varepsilon \end{aligned}$$

for $1 \leq r \leq s$ and $1 \leq j \leq t$, as desired. ■

Proposition 6.14. Let θ be a nondegenerate skew symmetric real $d \times d$ matrix. Let $n \in \mathbf{N}$, let $\omega = \exp(2\pi i/n)$, let $1 \leq k \leq d$, and, following Notation 5.1, let $\alpha: A_\theta \rightarrow A_\theta$ the unique automorphism satisfying $\alpha(u_k) = \omega u_k$ and $\alpha(u_r) = u_r$ for $r \neq k$. Then the action of \mathbf{Z}_n generated by α has the tracial Rokhlin property.

Proof. Let τ be the unique tracial state on A_θ (Theorem 5.9). We will show that for every $\varepsilon > 0$ and every finite subset $F \subset A_\theta$, there are mutually orthogonal projections $e_0, e_1, \dots, e_{n-1} \in A_\theta$ such that:

- (1) $\|\alpha(e_j) - e_{j+1}\| < \varepsilon$ for $0 \leq j \leq n-1$.
- (2) $\|e_j a - a e_j\| < \varepsilon$ for $0 \leq j \leq n-1$ and $a \in F$.
- (3) $1 - n\tau(e_0) < \varepsilon$.

We first argue this is enough to deduce the tracial Rokhlin property. We must prove Conditions (3) and (4) in Definition 1.1. We first recall that if $p, q \in A_\theta$ are projections with $\tau(p) < \tau(q)$, then $p \lesssim q$. This follows from Theorems 6.1 and 7.1 of [52], or from Theorems 1.4(d) and 1.5 of [5]. Also, A_θ has Property (SP) by Theorem 1.4(b) of [5]. If now $N \in \mathbf{N}$ and a nonzero positive element $x \in A_\theta$ are given, then we may use Property (SP) to find a nonzero projection $p \in \overline{xAx}$. Let $e = \sum_{j=0}^{n-1} e_j$. Assume, as we clearly may, that $\varepsilon < 1$. Then $\alpha(e_j) \sim e_{j+1}$. So $\tau(1 - e) = 1 - n\tau(e_0) < \varepsilon$, whence $\tau(e_0) > \frac{1}{n}(1 - \varepsilon)$. If

$$\varepsilon < \min\left(\frac{1}{2}, \frac{1}{2nN}, \tau(p)\right),$$

then $\tau(1 - e) < \varepsilon$ implies $\tau(1 - e) < \tau(p)$ and $N\tau(1 - e) < \tau(e_0)$, so that Conditions (3) and (4) of Definition 1.1 follow from the comparison results above.

Now we prove Conditions (1)–(3) at the beginning of the proof. Let the notation be as in Theorem 6.1 and Notation 6.2. Let $\varepsilon > 0$. Choose and fix $\eta_0 \in \mathbf{R} \setminus \mathbf{Q}$. Choose $\varepsilon_1 > 0$ such that whenever a_0, a_1, \dots, a_{n-1} are elements of a unital C*-algebra D with

$$\|a_j a_r - \delta_{j,r} a_j\| < \varepsilon_1 \quad \text{and} \quad \|a_j^* - a_j\| < \varepsilon_1$$

for $0 \leq j, r \leq n - 1$, then there are mutually orthogonal projections

$$q_0, q_1, \dots, q_{n-1} \in D$$

such that $\|q_j - a_j\| < \frac{1}{3}n^{-1}\varepsilon$ for $0 \leq j \leq n - 1$. (For example, apply Definition 2.2 and Lemma 2.3 of [5] with the finite dimensional C*-algebra B taken to be \mathbf{C}^{n+1} , using in addition the element $a_n = 1 - \sum_{j=0}^{n-1} a_j$.) Let $p_0, p_1, \dots, p_{n-1} \in A_{\eta_0}$ be the projections e_0, e_1, \dots, e_{n-1} of Lemma 6.4 for η_0 in place of η and $\frac{1}{3}\varepsilon$ in place of ε . Because the rotation algebras form a continuous field over S^1 with section algebra A (Theorem 6.1), we may choose $c_0, c_1, \dots, c_{n-1} \in A$ such that $\text{ev}_{\eta_0}(c_j) = p_j$ for $0 \leq j \leq n - 1$, and we can furthermore find $\delta_0 > 0$ such that $|\exp(2\pi i\eta) - \exp(2\pi i\eta_0)| < \delta_0$ implies

$$\|\text{ev}_\eta(c_j)\text{ev}_\eta(c_r) - \delta_{j,r}\text{ev}_\eta(c_j)\| < \varepsilon_1 \quad \text{and} \quad \|\text{ev}_\eta(c_j)^* - \text{ev}_\eta(c_j)\| < \varepsilon_1$$

for $0 \leq j, r \leq n - 1$. Let $V \subset S^1$ be an open set such that $\exp(2\pi i\eta_0) \in V$ and such that $\zeta \in \overline{V}$ implies $|\zeta - \exp(2\pi i\eta_0)| < \delta_0$. Letting $c_j|_{\overline{V}}$ denote the restriction of c_j , regarded as a section, to \overline{V} , we get

$$\|(c_j|_{\overline{V}})(c_r|_{\overline{V}}) - \delta_{j,r}c_j|_{\overline{V}}\| < \varepsilon_1 \quad \text{and} \quad \|(c_j|_{\overline{V}})^* - c_j|_{\overline{V}}\| < \varepsilon_1$$

for $0 \leq j, r \leq n - 1$, so that there are mutually orthogonal projections

$$q_0, q_1, \dots, q_{n-1} \in \Gamma(\overline{V})$$

such that $\|q_j - c_j|_{\overline{V}}\| < \frac{1}{3}n^{-1}\varepsilon$ for $0 \leq j \leq n - 1$. Since the restriction map $A = \Gamma(S^1) \rightarrow \Gamma(\overline{V})$ is surjective, there exist $b_0, b_1, \dots, b_{n-1} \in A$ such that $b_j|_{\overline{V}} = q_j$ for $0 \leq j \leq n - 1$.

Let the generators of A be as in Theorem 6.1, and let $\beta \in \text{Aut}(A)$ be the unique automorphism such that

$$\beta(v) = \omega v, \quad \beta(w) = w, \quad \text{and} \quad \beta(z) = z.$$

Let $\beta_\eta \in \text{Aut}(A_\eta)$ be defined by $\beta_\eta(v_\eta) = \omega v_\eta$ and $\beta_\eta(w_\eta) = w_\eta$. Then $\text{ev}_\eta \circ \beta = \beta_\eta \circ \text{ev}_\eta$. Since β sends continuous sections to continuous sections, there is an open set $U_0 \subset V$ such that $\eta_0 \in U_0$ and if $\eta \in U_0$ then for $0 \leq j \leq n - 1$ and with $b_n = b_0$,

$$\|\beta_\eta(b_j(\eta)) - b_{j+1}(\eta)\| \quad \text{and} \quad \|\beta_{\eta_0}(b_j(\eta_0)) - b_{j+1}(\eta_0)\|$$

differ by less than $\frac{1}{3}\varepsilon$. For such η we have $b_j(\eta) = q_j(\eta)$, so, using $c_j(\eta_0) = p_j$ and $\beta_{\eta_0}(p_j) = p_{j+1}$ at the second last step,

$$\begin{aligned} \|\beta_\eta(q_j(\eta)) - q_{j+1}(\eta)\| &< \|\beta_{\eta_0}(q_j(\eta_0)) - q_{j+1}(\eta_0)\| + \frac{1}{3}\varepsilon \\ &< \|q_j - c_j|_{\overline{V}}\| + \|q_{j+1} - c_{j+1}|_{\overline{V}}\| + \|\beta_{\eta_0}(c_j(\eta_0)) - c_{j+1}(\eta_0)\| + \frac{1}{3}\varepsilon \\ &< \frac{1}{3}n^{-1}\varepsilon + \frac{1}{3}n^{-1}\varepsilon + \frac{1}{3}\varepsilon \leq \varepsilon \end{aligned}$$

for $0 \leq j \leq n - 1$.

Using Lemma 6.3, choose an open set $U \subset U_0$ such that $\eta_0 \in U$ and if $\eta \in U$ then for $0 \leq j \leq n - 1$ we have $|\tau_\eta(q_j(\eta)) - \tau_{\eta_0}(q_j(\eta_0))| < \frac{1}{3}n^{-1}\varepsilon$.

Apply Lemma 6.13 with θ, n, k, η_0, U , and F as given, with $\min(\varepsilon, \delta)$ in place of ε , and with $S = \{q_0, q_1, \dots, q_{n-1}\}$. We obtain $\eta \in (\mathbf{R} \setminus \mathbf{Q}) \cap U$ and

$$l = (l_1, l_2, \dots, l_d) \in \mathbf{Z}^d \quad \text{and} \quad m = (m_1, m_2, \dots, m_d) \in \mathbf{Z}^d.$$

Set

$$x = u_1^{l_1} u_2^{l_2} \cdots u_d^{l_d} \quad \text{and} \quad y = u_1^{m_1} u_2^{m_2} \cdots u_d^{m_d},$$

so that $yx = \exp(2\pi i\eta)xy$. Let $\varphi: A_\eta \rightarrow A_\theta$ be the homomorphism such that $\varphi(v_\eta) = x$ and $\varphi(w_\eta) = y$, and set $e_j = \varphi(q_j(\eta))$ for $0 \leq j \leq n - 1$. We verify Conditions (1), (2), and (3) at the beginning of the proof for this choice of e_0, e_1, \dots, e_{n-1} .

We do Condition (1). Because $l_k = 1 \pmod{n}$ and $m_k = 0 \pmod{n}$, we have $\alpha(x) = \omega x$ and $\alpha(y) = y$. It follows that $\alpha \circ \varphi = \varphi \circ \beta_\eta$. Therefore

$$\|\alpha(e_j) - e_{j+1}\| \leq \|\beta_\eta(q_j(\eta)) - q_{j+1}(\eta)\| < \varepsilon$$

for $0 \leq j \leq n - 1$, as desired.

Condition (2) is immediate from Part (3) of Lemma 6.13.

Finally, we check Condition (3). By uniqueness of the tracial states, we have $\tau \circ \varphi = \tau_\eta$. Therefore, using the choice of U at the second step and $\|q_j(\eta_0) - p_j\| < \frac{1}{3}n^{-1}\varepsilon$ at the third step, we get

$$\tau(e_j) = \tau_\eta(q_j(\eta)) > \tau_{\eta_0}(q_j(\eta_0)) - \frac{1}{3}n^{-1}\varepsilon > \tau_{\eta_0}(p_j) - \frac{2}{3}n^{-1}\varepsilon.$$

Therefore

$$1 - n\tau(e_0) < 1 - n\tau(p_0) + \frac{2}{3}\varepsilon < \frac{1}{3}\varepsilon + \frac{2}{3}\varepsilon = \varepsilon.$$

This completes the proof of (3). ■

As Hanfeng Li pointed out, the following lemma also holds when θ is degenerate.

Lemma 6.15. Let θ be a nondegenerate skew symmetric real $d \times d$ matrix. Let $n \in \mathbf{N}$, let $\omega = \exp(2\pi i/n)$, let $1 \leq l \leq d$, and, following Notation 5.1, let $\alpha: A_\theta \rightarrow A_\theta$ the unique automorphism satisfying $\alpha(u_l) = \omega u_l$ and $\alpha(u_k) = u_k$ for $k \neq l$. Let $B \in \mathrm{GL}_d(\mathbf{Q})$ be the matrix $B = \mathrm{diag}(1, \dots, 1, n, 1, \dots, 1)$, where n is in the l -th position. Then the fixed point algebra A_θ^α is isomorphic to $A_{B^t\theta B}$.

Proof. We observe that

$$(B^t\theta B)_{j,k} = \begin{cases} n\theta_{j,k} & j = l \text{ or } k = l \\ \theta_{j,k} & \text{otherwise} \end{cases}.$$

(Note that $(B^t\theta B)_{l,l} = \theta_{l,l} = 0$.) Moreover, $B^t\theta B$ is nondegenerate by Lemma 5.8. Therefore

$$D = C^*(u_1, \dots, u_{l-1}, u_l^n, u_{l+1}, \dots, u_d) \subset A_\theta$$

is isomorphic to $A_{B^t\theta B}$.

We claim that $A_\theta^\alpha = D$. That $D \subset A_\theta^\alpha$ is clear. For the reverse inclusion, define $E: A_\theta \rightarrow A_\theta^\alpha$ by $E(a) = \frac{1}{n} \sum_{j=0}^{n-1} \alpha^j(a)$. Then E is a surjective continuous linear map, so it suffices to show that $E(u_1^{m_1} u_2^{m_2} \cdots u_d^{m_d}) \in D$ for all $m = (m_1, m_2, \dots, m_d) \in \mathbf{Z}^d$. If m_l is divisible by n then $u_1^{m_1} u_2^{m_2} \cdots u_d^{m_d}$ is a fixed point of E and is in D , and otherwise $E(u_1^{m_1} u_2^{m_2} \cdots u_d^{m_d}) = 0 \in D$. ■

Corollary 6.16. Let θ be a nondegenerate skew symmetric real $d \times d$ matrix. Let $n \in \mathbf{N}$, let $1 \leq l \leq d$, and let

$$B = \mathrm{diag}(1, \dots, 1, n, 1, \dots, 1) \in \mathrm{GL}_d(\mathbf{Q}),$$

where n is in the l -th position. Then $A_{B^t\theta B}$ has tracial rank zero if and only if A_θ has tracial rank zero.

Proof. The C^* -algebra A_θ has cancellation of projections, by Theorems 6.1 and 7.1 of [52], or by [27] and Theorems 1.4(d) and 1.5 of [5]. The corollary therefore follows from Lemma 6.15, Proposition 6.14, Corollary 6.9, Corollary 4.6, and Corollary 4.7. ■

7. DIRECT LIMIT DECOMPOSITION FOR SIMPLE NONCOMMUTATIVE TORUSES

In this section, we use the results of the previous two sections to prove that every simple higher dimensional noncommutative torus is an AT algebra. (See the introduction to Section 5 for a discussion of previous work on this problem.)

The following result is essentially Corollary 6.6 of [35].

Proposition 7.1. Let α be a nondegenerate skew symmetric real bicharacter on \mathbf{Z}^n . Suppose that $A_{\alpha|_{\mathbf{Z}^{n-1} \times \{0\}}}$ is a simple AT algebra with real rank zero. Then A_α is a simple AT algebra with real rank zero.

Proof. Let $\beta = \alpha|_{\mathbf{Z}^{n-1} \times \{0\}}$. We note that

$$K_0(A_\beta) \cong K_1(A_\beta) \cong \mathbf{Z}^{2^{n-1}}$$

by Lemma 5.5 and by repeated application of the Pimsner-Voiculescu exact sequence [48]. In particular, both groups are finitely generated. Further write $A_\alpha = C^*(\mathbf{Z}, A_\beta, \varphi)$ as in Lemma 5.5, with φ homotopic to the identity. Thus, in the notation of [35] (see the introduction to [35]), $\varphi \in \text{HInn}(A_\beta)$. So the proof of Corollary 6.5 of [35] shows that the hypotheses of Theorem 6.4 of [35] hold. We know from Lemma 5.9 that $A_\alpha = C^*(\mathbf{Z}, A_\beta, \varphi)$ has a unique tracial state. Therefore Theorem 6.4 of [35] implies that $A_\alpha = C^*(\mathbf{Z}, A_\beta, \varphi)$ is a simple AT algebra with real rank zero. ■

Lemma 7.2. The group $\text{GL}_d(\mathbf{Q})$ is generated as a group by $\text{GL}_d(\mathbf{Z})$ and all matrices of the form $\text{diag}(1, \dots, 1, n, 1, \dots, 1)$, where $n \in \mathbf{N}$ is nonzero and is in an arbitrary position.

Proof. Let G be the subgroup of $\text{GL}_d(\mathbf{Q})$ generated by $\text{GL}_d(\mathbf{Z})$ and the matrices $\text{diag}(1, \dots, 1, n, 1, \dots, 1)$. It suffices to show that G contains all of the following three kinds of elementary matrices:

$$E_j^{(1)}(r) = \text{diag}(1, \dots, 1, r, 1, \dots, 1),$$

where $r \in \mathbf{Q} \setminus \{0\}$ and is the j -th diagonal entry in the matrix; the transposition matrix $E_{j,k}^{(2)}$, for $1 \leq j < k \leq d$, which acts on the standard basis vectors by

$$E_{j,k}^{(2)}(\delta_l) = \begin{cases} \delta_l & l \neq j, k \\ \delta_k & l = j \\ \delta_j & l = k \end{cases};$$

and the matrix $E_{j,k}^{(3)}(r)$ for $1 \leq j, k \leq n$ with $j \neq k$ and $r \in \mathbf{Q}$, given by

$$E_{j,k}^{(3)}(r)(\delta_l) = \begin{cases} \delta_l & l \neq k \\ \delta_k + r\delta_j & l = k \end{cases}.$$

If $r = (-1)^m p/q$ with $m = 0$ or $m = 1$ and with p and q positive integers, then

$$E_j^{(1)}(r) = E_j^{(1)}((-1)^m) E_j^{(1)}(p) [E_j^{(1)}(q)]^{-1},$$

where the first factor is in $\text{GL}_d(\mathbf{Z})$ and $E_j^{(1)}(p)$ and $E_j^{(1)}(q)$ are among the other generators of G . The matrix $E_{j,k}^{(2)}$ is already in $\text{GL}_d(\mathbf{Z})$. For $E_{j,k}^{(3)}(r)$, we may conjugate by a permutation matrix, which is in $\text{GL}_d(\mathbf{Z})$, and split off as a direct

summand a $(d-2) \times (d-2)$ identity matrix, and thus reduce to the case $d=2$, $j=1$, and $k=2$. Write $r=p/q$ with $p \in \mathbf{Z}$ and $q \in \mathbf{N}$. Then the factorization

$$E_{1,2}^{(3)}(r) = \begin{pmatrix} 1 & p/q \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} q & 0 \\ 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & p \\ 0 & 1 \end{pmatrix} \begin{pmatrix} q & 0 \\ 0 & 1 \end{pmatrix}$$

shows that $E_{1,2}^{(3)}(r) \in G$. ■

Corollary 7.3. Let θ be a nondegenerate skew symmetric real $d \times d$ matrix. Let $B \in \mathrm{GL}_d(\mathbf{Q})$. Then $A_{B^t\theta B}$ has tracial rank zero if and only if A_θ has tracial rank zero.

Proof. Combine Lemma 7.2, Remark 5.2, and Corollary 6.16. ■

Lemma 7.4. Let θ be a nondegenerate skew symmetric real $d \times d$ matrix, with $d > 2$. Suppose that there is no subgroup H of \mathbf{Z}^d of rank $d-1$ such that $\theta|_H$ (in the sense of Remark 5.4) is nondegenerate. Let $r < d-1$ be the maximal rank of a proper subgroup H of \mathbf{Z}^d such that $\theta|_H$ is nondegenerate. Then there exists $B \in \mathrm{GL}_d(\mathbf{Q})$ such that $B^t\theta B$ has the block form

$$B^t\theta B = \begin{pmatrix} \rho_{1,1} & \rho_{1,2} \\ (\rho_{1,2})^t & \rho_{2,2} \end{pmatrix},$$

with $\rho_{1,1}$ and $\rho_{2,2}$ nondegenerate skew symmetric real $r \times r$ and $(d-r) \times (d-r)$ matrices, and where all the entries of $\rho_{1,2}$ are in \mathbf{Z} .

Proof. Let $H \subset \mathbf{Z}^d$ be a subgroup of rank r such that $\theta|_H$ is nondegenerate. Since θ is not rational, clearly $r \geq 2$. Let (v_1, v_2, \dots, v_r) be a basis for H over \mathbf{Z} . Choose $v_{r+1}, \dots, v_d \in \mathbf{Z}^d$ such that (v_1, v_2, \dots, v_d) is a basis for \mathbf{Q}^d over \mathbf{Q} . For $r+1 \leq k \leq d$, by hypothesis $\theta|_{H+\mathbf{Z}v_k}$ is degenerate. By Lemma 5.7, there exists $x_k \in \mathrm{span}_{\mathbf{Q}}(H \cup \{v_k\}) \setminus \{0\}$ such that $\langle x_k, \theta y \rangle \in \mathbf{Q}$ for all $y \in \mathrm{span}_{\mathbf{Q}}(H \cup \{v_k\})$. Since $\theta|_H$ is nondegenerate, we have $x_k \notin \mathrm{span}_{\mathbf{Q}}(H)$. Therefore $v_k \in \mathrm{span}_{\mathbf{Q}}(H \cup \{x_k\})$. It follows that

$$v_{r+1}, \dots, v_d \in \mathrm{span}_{\mathbf{Q}}(v_1, v_2, \dots, v_r, x_{r+1}, \dots, x_d),$$

so that $(v_1, v_2, \dots, v_r, x_{r+1}, \dots, x_d)$ is a basis for \mathbf{Q}^d . By construction, we have $\langle x_k, \theta v_l \rangle \in \mathbf{Q}$ for $1 \leq l \leq r$ and $r+1 \leq k \leq d$. Choose $N \in \mathbf{Z} \setminus \{0\}$ such that $N\langle x_k, \theta v_l \rangle \in \mathbf{Z}$ for $1 \leq l \leq r$ and $r+1 \leq k \leq d$.

Let $B \in \mathrm{GL}_d(\mathbf{Q})$ be the matrix whose action on the standard basis vectors is

$$B\delta_k = \begin{cases} v_k & 1 \leq k \leq r \\ Nx_k & r+1 \leq k \leq d \end{cases}.$$

Then for $1 \leq l \leq r$ and $r+1 \leq k \leq d$, we have

$$\langle \delta_k, B^t\theta B\delta_l \rangle = N\langle x_k, \theta v_l \rangle \in \mathbf{Z}.$$

Since $B^t\theta B$ is skew symmetric, this shows that it has a block decomposition of the required form. It is immediate to check that the two diagonal blocks must be nondegenerate, since otherwise $B^t\theta B$ would be degenerate, contradicting Lemma 5.8. ■

We will use below, and also on several later occasions, the following consequence of H. Lin's classification theorem [42]. An AH algebra is a direct limit of finite direct sums of corners of homogeneous C*-algebras whose primitive ideal spaces are finite CW complexes. See, for example, the statement of Theorem 4.6 of [17],

except that we omit the restrictions there on the type of CW complexes which may appear; or see 2.5 of [37]. Also, see Definition 8.7 and Remark 8.8 below for a careful statement of what it means to satisfy the Universal Coefficient Theorem.

Lemma 7.5. Let A be a simple infinite dimensional separable unital nuclear C^* -algebra with tracial rank zero and which satisfies the Universal Coefficient Theorem. Then A is a simple AH algebra with real rank zero and no dimension growth. If $K_*(A)$ is torsion free, then A is an AT algebra. If, in addition, $K_1(A) = 0$, then A is an AF algebra.

Proof. Theorems 6.11 and 6.13 of [38] show that $K_0(A)$ is weakly unperforated and is a Riesz group. We now apply Theorem 4.20 of [17] to find a simple unital AH algebra B with real rank zero and no dimension growth whose ordered scaled K-theory is the same as that of A . If $K_*(A)$ is torsion free, we claim that there is a simple AT algebra B with real rank zero whose ordered scaled K-theory is the same as that of A . To prove this, note that $K_0(A)$ can't be \mathbf{Z} because A has real rank zero; then we apply the proof of Theorem 8.3 of [14]. (As noted in the introduction to [14], the part of the order involving K_1 is irrelevant in the simple case.) We can certainly take the groups in the direct limit decomposition to be torsion free, so that the proof shows that all the algebras in the direct system constructed there may be taken to have primitive ideal space the circle or a point. Then Theorem 4.3 of [14] shows they may all be taken to have primitive ideal space the circle. This gives the required AT algebra B . Finally, if in addition $K_1(A) = 0$, following [12] we may find a simple AF algebra B whose ordered scaled K-theory is the same as that of A .

Proposition 2.6 of [37] (with \mathcal{C} as defined in 2.5 of [37]) implies that simple AH algebras with real rank zero and no dimension growth have tracial rank zero. In particular, B has tracial rank zero. So the classification theorem for C^* -algebras with tracial rank zero, Theorem 5.2 of [42], implies that $A \cong B$. ■

Theorem 7.6. Let θ be a nondegenerate skew symmetric real $d \times d$ matrix, with $d \geq 2$. Then A_θ is a simple AT algebra with real rank zero, and in particular has tracial rank zero.

Proof. We prove this by induction on d . The first part of the conclusion is true for $d = 2$ by the Elliott-Evans Theorem [15], and tracial rank zero follows from Proposition 2.6 of [37] (with \mathcal{C} as defined in 2.5 of [37]). Suppose d is given, and the theorem is known for all skew symmetric real $k \times k$ matrices with $k < d$. Let θ be a nondegenerate skew symmetric real $d \times d$ matrix. There are two cases.

First, suppose that there is a subgroup H_0 of \mathbf{Z}^d of rank $d - 1$ such that $\theta|_{H_0}$ (in the sense of Remark 5.4) is nondegenerate. Set

$$H = \{x \in \mathbf{Z}^d : \text{There is } n \in \mathbf{Z} \text{ such that } nx \in H_0\}.$$

Then H is also a subgroup of \mathbf{Z}^d of rank $d - 1$, and $\theta|_{H_0}$ is also nondegenerate. Moreover, \mathbf{Z}^d/H is torsion free and therefore isomorphic to \mathbf{Z} , from which it follows that the quotient map splits. Thus there is an isomorphism $\mathbf{Z}^d \rightarrow \mathbf{Z}^d$ which sends H isomorphically onto $\mathbf{Z}^{d-1} \oplus \{0\} \subset \mathbf{Z}^d$. Accordingly, we may assume that $H = \mathbf{Z}^{d-1} \oplus \{0\}$. By the induction hypothesis, $A_{\theta|_H}$ is a simple AT algebra with real rank zero. So Proposition 7.1 implies that A_θ is a simple AT algebra with real rank zero.

Now assume there is no such subgroup H_0 of rank $d - 1$. Let B be as in Lemma 7.4, with

$$B^t \theta B = \begin{pmatrix} \rho_{1,1} & \rho_{1,2} \\ (\rho_{1,2})^t & \rho_{2,2} \end{pmatrix},$$

and where in particular all the entries of $\rho_{1,2}$ are in \mathbf{Z} . Then $A_{B^t \theta B} \cong A_\rho$ for

$$\rho = \begin{pmatrix} \rho_{1,1} & 0 \\ 0 & \rho_{2,2} \end{pmatrix}.$$

Since ρ , $\rho_{1,1}$, and $\rho_{2,2}$ are all nondegenerate, it easily follows that $A_\rho \cong A_{\rho_{1,1}} \otimes A_{\rho_{2,2}}$. By the induction hypothesis, both $A_{\rho_{1,1}}$ and $A_{\rho_{2,2}}$ are simple AT algebras with real rank zero. Therefore $A_{\rho_{1,1}} \otimes A_{\rho_{2,2}}$ is a simple direct limit, with no dimension growth, of homogeneous C*-algebras. Since it has a unique tracial state, Theorems 1 and 2 of [4] imply that A_ρ has stable rank one and real rank zero. This algebra has weakly unperforated K-theory by Theorem 6.1 of [52]. (Actually, this is true for any direct limit of the type at hand.) It now follows from Theorem 4.6 of [40] that $A_{\rho_{1,1}} \otimes A_{\rho_{2,2}}$ has tracial rank zero. So Corollary 7.3 shows that A_θ has tracial rank zero. Using Theorem 1.17 of [57] (see the preceding discussion for the definition of \mathcal{N}), it follows from Lemma 5.5 that A_θ satisfies the Universal Coefficient Theorem. Clearly A_θ is separable and nuclear. Since

$$K_0(A_\beta) \cong K_1(A_\beta) \cong \mathbf{Z}^{2^{n-1}}$$

by Lemma 5.5 and by repeated application of the Pimsner-Voiculescu exact sequence [48], Lemma 7.5 implies that A_θ is an AT algebra. ■

We note that one could use the earlier Theorem 3.11 of [16] to show that $A_{\rho_{1,1}} \otimes A_{\rho_{2,2}}$ is an AT algebra with real rank zero, from which it follows that this algebra has tracial rank zero. The use of H. Lin's classification theorem, Theorem 5.2 of [42], remains essential, because we can only relate tracial rank zero to crossed products and fixed point algebras of actions by finite cyclic groups, not the property of being an AT algebra or even an AH algebra.

Remark 7.7. Since the paper [27] remains unpublished, it is worth pointing out that the proof of Theorem 7.6 does not actually depend on this paper. In the proof of Lemma 6.4, we need to know that the ordinary irrational rotation algebras have real rank zero, but this follows from Remark 6 of [15]. In the proof of Proposition 6.14, we need to know that traces determine order on projections in A_θ whenever A_θ is simple. The proof of this in [5] does not rely on [27], and in any case an independent proof (valid whenever θ is not purely rational) is contained in [52]. And in the application of Theorem 6.4 of [35] in the proof of Proposition 7.1, we use the fact that A_α has a unique tracial state, rather than real rank zero, to show that Kishimoto's conditions hold.

One might also hope to prove Lemma 6.14 using Theorem 8.2, as is done for the noncommutative Fourier transform in Lemma 8.4 and Proposition 8.6. However, Theorem 8.2 requires that one know ahead of time that the algebra involved has tracial rank zero.

Recall that the opposite algebra A^{op} of a C*-algebra A is the algebra A with the multiplication reversed but all other operations, including the scalar multiplication, the same.

Corollary 7.8. Let θ be a nondegenerate skew symmetric real $d \times d$ matrix, with $d \geq 2$. Then $(A_\theta)^{\text{op}} \cong A_\theta$.

Proof. Every simple AT algebra A with real rank zero is isomorphic to its opposite algebra, because the ordered K-theory of A^{op} is the same as the ordered K-theory of A . ■

As far as we know, it is unknown whether $(A_\theta)^{\text{op}} \cong A_\theta$ for general degenerate θ .

Remark 7.9. In [55], certain Morita equivalences between higher dimensional non-commutative toruses were exhibited. Combining H. Lin's classification theorem for simple nuclear C*-algebras with tracial rank zero and the computation of the range of the trace on $K_0(A_\theta)$ in [13], one should be able to completely determine the Morita equivalence classes of simple higher dimensional noncommutative toruses. We do not carry this out here.

8. THE NONCOMMUTATIVE FOURIER TRANSFORM

For $\theta \in \mathbf{R}$ let A_θ be the ordinary rotation algebra, the universal C*-algebra generated by unitaries u and v satisfying $vu = \exp(2\pi i\theta)uv$. The group $\text{SL}_2(\mathbf{Z})$ acts on A_θ by sending the matrix

$$n = \begin{pmatrix} n_{1,1} & n_{1,2} \\ n_{2,1} & n_{2,2} \end{pmatrix}$$

to the automorphism determined by

$$\alpha_n(u) = \exp(\pi i n_{1,1} n_{2,1} \theta) u^{n_{1,1}} v^{n_{2,1}} \quad \text{and} \quad \alpha_n(v) = \exp(\pi i n_{1,2} n_{2,2} \theta) u^{n_{1,2}} v^{n_{2,2}}.$$

For fixed θ , the actions of finite cyclic subgroups of $\text{SL}_2(\mathbf{Z})$ are classified and their fixed point algebras are found in [18]; the fixed point algebras are described in terms of generators and relations in [8] and [22]. There are essentially only four actions: an action of \mathbf{Z}_2 generated by the flip automorphism

$$u \mapsto u^* \quad \text{and} \quad v \mapsto v^*,$$

an action of \mathbf{Z}_3 generated by the automorphism

$$u \mapsto e^{-\pi i \theta} u^* v \quad \text{and} \quad v \mapsto u^*,$$

an action of \mathbf{Z}_4 generated by the noncommutative Fourier transform

$$u \mapsto v \quad \text{and} \quad v \mapsto u^*,$$

and an action of \mathbf{Z}_6 generated by the automorphism

$$u \mapsto v \quad \text{and} \quad v \mapsto e^{-\pi i \theta} u^* v.$$

One can define a few other automorphisms of the same order by changing the scalar factors in the formulas above, but one does not get anything essentially new. See Proposition 21 of [23].

Here, we will primarily be concerned with the crossed products, although, as we will see, for θ irrational the crossed products are all Morita equivalent to the corresponding fixed point algebras. The most is known about the crossed product by the flip: its (unordered) K-theory has been computed in [36], and the crossed product has been proved to be an AF algebra [10]. The next best understood case is that of the noncommutative Fourier transform, which has been intensively studied in a series of papers culminating in [61]. It is proved there that for “most” irrational θ , the crossed product of A_θ by the noncommutative Fourier transform

has tracial rank zero in the sense of [37], and that for “most” of those values of θ , it is in addition an AF algebra. There are three independent parts of the proof that this crossed product is AF for “most” θ : the proof that it satisfies the Universal Coefficient Theorem, the proof that it has tracial rank zero, and the computation of the K-theory. In [60] the K-theory computation is done for “most” θ , and in [61] the crossed product is shown to have tracial rank zero for “most” θ , and to satisfy the Universal Coefficient Theorem for all θ . We prove here, using completely different methods, that the crossed product and the fixed point algebra have tracial rank zero for all irrational θ , not just “most” θ . Instead of the heavy use of theta functions in [61], we prove that the action of \mathbf{Z}_4 has the tracial Rokhlin property. We conclude that the crossed product is a simple AH algebra with real rank zero for all irrational θ . To show that the crossed product is AF for all irrational θ requires in addition an improvement on the K-theory calculation. This will be done elsewhere, in joint work with Wolfgang Lück and Sam Walters.

The same methods show that the other three finite cyclic group actions also have the tracial Rokhlin property, and that their crossed products also satisfy the Universal Coefficient Theorem. These crossed products are therefore also simple AH algebras with real rank zero, for all irrational θ . Combining this with the K-theory computation in [36], one obtains a new proof that for irrational θ the crossed product by the flip is AF. This computation is subsumed by the work of the next section, in which we show that the crossed product of any simple higher dimensional noncommutative torus by the analog of the flip is AF. For the actions of \mathbf{Z}_3 and of \mathbf{Z}_6 , we don’t know enough about the K-theory to conclude that any of the crossed products are AF.

In this section, we do the case of the noncommutative Fourier transform in detail. In the next section, we give a brief description for the actions of \mathbf{Z}_3 and \mathbf{Z}_6 .

We start with a general method for proving that an action has the tracial Rokhlin property when the algebra has tracial rank zero and a unique tracial state. Most of the work is contained in the first lemma. In Theorem 8.2, we give a useful further weakening of the hypotheses.

As Masaki Izumi has pointed out, we could prove outerness of the actions we are interested in by using the fact that the corresponding actions on the factor obtained from the trace representation are outer. We have decided to keep the original proof because we hope it points the way to generalizations of the methods, and perhaps to the right version of the tracial Rokhlin property for actions on C*-algebras with few projections.

Lemma 8.1. Let A be an infinite dimensional simple unital C*-algebra with tracial rank zero and with a unique tracial state τ . Let $\alpha \in \text{Aut}(A)$ satisfy $\alpha^n = \text{id}_A$. Suppose that for every finite set $F \subset A$ and every $\varepsilon > 0$ there are positive elements $a_0, a_1, \dots, a_{n-1} \in A$ with $0 \leq a_j \leq 1$ such that:

- (1) $a_j a_k = 0$ for $j \neq k$.
- (2) $\|\alpha(a_j) - a_{j+1}\| < \varepsilon$ for $0 \leq j \leq n-2$.
- (3) $\|a_j c - c a_j\| < \varepsilon$ for $0 \leq j \leq n-1$ and all $c \in F$.
- (4) $\tau\left(1 - \sum_{j=0}^{n-1} a_j\right) < \varepsilon$.

Then the action of \mathbf{Z}_n generated by α has the tracial Rokhlin property.

Proof. We verify the conditions of Definition 1.1. Note that $\tau \circ \alpha = \tau$. Let $F \subset A$ be a finite set, let $\varepsilon > 0$, let $N \in \mathbf{N}$, and let $x \in A$ be a nonzero positive element.

Without loss of generality F is α -invariant and $\|a\| \leq 1$ for all $a \in F$; also, $\varepsilon < 1$. Since A has real rank zero (Theorem 2.2), there is a nonzero projection $q \in \overline{xAx}$. Since also A is infinite dimensional, simple, and unital, Theorem 1.1(i) of [62] provides a nonzero projection $q_0 \in A$ such that

$$\tau(q_0) < \min\left(\frac{\tau(q)}{2n}, \frac{1}{8n}, \frac{1}{4Nn^2}\right).$$

By Lemma 2.3 of [5], applied to $B = \mathbf{C}^{n+1}$ (see Definition 2.2 of [5]), there is $\delta > 0$ such that whenever D is a C^* -algebra and $p_1, p_2, \dots, p_n \in D$ are projections such that $\|p_j p_k\| < \delta$ for $1 \leq j, k \leq n$ with $j \neq k$, then there are mutually orthogonal projections $q_1, q_2, \dots, q_n \in D$ such that $\|q_k - p_k\| < \frac{1}{10}\varepsilon$ for $1 \leq k \leq n$. Set

$$\varepsilon_0 = \min\left(\frac{\delta}{5}, \frac{1}{8n(n+4)}, \frac{1}{4Nn^2(n+4)}, \frac{\tau(q)}{2n(n+4)}\right).$$

By Proposition 2.1, there is a projection $p \in A$ and a finite dimensional unital subalgebra $E \subset pAp$ such that:

- (1) $\|pa - ap\| < \frac{1}{10}\varepsilon$ for all $a \in F$.
- (2) For every $a \in F$ there exists $b \in E$ such that $\|pap - b\| < \frac{1}{10}\varepsilon$.
- (3) $1 - p$ is Murray-von Neumann equivalent to a projection in q_0Aq_0 .

Let $p^{(1)}, p^{(2)}, \dots, p^{(s)}$ be the minimal central projections of E , write $p^{(l)}Ep^{(l)} = M_{r(l)}$ with $r(l) \in \mathbf{N}$, and let $\{p_{j,k}^{(l)} : 1 \leq j, k \leq r(l)\}$ be a system of matrix units for $p^{(l)}Ep^{(l)}$. Define $P : pAp \rightarrow pAp$ by

$$P(a) = \sum_{l=1}^s \sum_{k=1}^{r(l)} p_{k,1}^{(l)} a p_{1,k}^{(l)}.$$

Then P is a (slightly nonstandard) conditional expectation from pAp onto the relative commutant $E' \cap pAp$ of E in pAp .

Define

$$S = \{p_{j,k}^{(l)} : 1 \leq l \leq s \text{ and } 1 \leq j, k \leq r(l)\},$$

which is a complete system of matrix units for E , with cardinality $\text{card}(S)$. We claim that if $a \in pAp$ then

$$\|a - P(a)\| < \text{card}(S) \max_{v \in S} \|va - av\|.$$

To see this, first observe that

$$\|p_{k,k}^{(l)} a p_{k,k}^{(l)} - p_{k,1}^{(l)} a p_{1,k}^{(l)}\| = \|p_{k,k}^{(l)} [a p_{k,1}^{(l)} - p_{k,1}^{(l)} a] p_{1,k}^{(l)}\| \leq \|a p_{k,1}^{(l)} - p_{k,1}^{(l)} a\|.$$

Using this, we estimate:

$$\begin{aligned} \|a - P(a)\| &\leq \sum_{l=1}^s \sum_{j,k=1}^{r(l)} \|p_{j,j}^{(l)} [a - P(a)] p_{k,k}^{(l)}\| \\ &\leq \sum_{l=1}^s \left(\sum_{j \neq k} \|p_{j,j}^{(l)} a p_{k,k}^{(l)}\| + \sum_{k=2}^{r(l)} \|p_{k,k}^{(l)} a p_{k,k}^{(l)} - p_{k,1}^{(l)} a p_{1,k}^{(l)}\| \right) \\ &\leq \sum_{l=1}^s \left(\sum_{j \neq k} \|p_{j,j}^{(l)} a - a p_{j,j}^{(l)}\| + \sum_{k=2}^{r(l)} \|p_{k,1}^{(l)} a - a p_{k,1}^{(l)}\| \right) \\ &\leq \text{card}(S) \max_{v \in S} \|va - av\|. \end{aligned}$$

This proves the claim.

Define continuous functions $g_1, g_2, g_3: \mathbf{R} \rightarrow [0, 1]$ by

$$g_j(t) = \begin{cases} 0 & t \leq \frac{1}{3}(j-1) \\ 3t & \frac{1}{3}(j-1) \leq t \leq \frac{1}{3}j \\ 1 & \frac{1}{3}j \leq t \end{cases}$$

for $j = 1, 2, 3$. Then $g_1g_2 = g_2$ and $g_2g_3 = g_3$. Using polynomial approximations to g_1, g_2 , and g_3 , choose $\varepsilon_1 > 0$ so small that whenever D is a C^* -algebra and $a, b \in D$ satisfy $0 \leq a, b \leq 1$ and

$$\|a - b\| < \max(n, 2 + \text{card}(S))\varepsilon_1,$$

then

$$\|g_1(a) - g_1(b)\| < \varepsilon_0, \quad \|g_2(a) - g_2(b)\| < \varepsilon_0, \quad \text{and} \quad \|g_3(a) - g_3(b)\| < \varepsilon_0.$$

Apply the hypothesis with $S \cup \{p\}$ in place of F and with $\min(\varepsilon_1, \frac{1}{16}\varepsilon_0^2)$ in place of ε , obtaining a_0, a_1, \dots, a_{n-1} .

We have $\|(pa_0p)p_{j,k}^{(l)} - p_{j,k}^{(l)}(pa_0p)\| < \varepsilon_1$ for all l, j , and k , whence

$$\|P(pa_0p) - pa_0p\| < \text{card}(S)\varepsilon_1.$$

Also,

$$\begin{aligned} \|a_0 - [(1-p)a_0(1-p) + pa_0p]\| &\leq \|pa_0(1-p)\| + \|(1-p)a_0p\| \\ &\leq 2\|a_0p - pa_0\| < 2\varepsilon_1. \end{aligned}$$

Set $c = (1-p)a_0(1-p) + P(pa_0p)$. It follows that

$$\|a_0 - c\| < [2 + \text{card}(S)]\varepsilon_1.$$

Evaluating functional calculus in the appropriate corners on the right, we have

$$g_2(c) = g_2((1-p)a_0(1-p)) + g_2(P(pa_0p)).$$

Therefore, by the choice of ε_1 , we have

$$\|g_2(a_0) - [g_2((1-p)a_0(1-p)) + g_2(P(pa_0p))]\| < \varepsilon_0.$$

With $b = g_2(P(pa_0p))$, which is equal to $g_2(c)p \in E' \cap pAp$, we then get

$$\|g_1(a_0)b - b\| = \|[g_1(a_0)g_2(c) - g_2(c)]p\| \leq 2\|g_2(c) - g_2(a_0)\| < 2\varepsilon_0.$$

Similarly, we have $\|bg_1(a_0) - b\| < 2\varepsilon_0$.

An inductive argument gives, for $0 \leq j \leq n-1$, the first inequality in the estimate

$$\|\alpha^j(a_0) - a_j\| < j\varepsilon_1 \leq n\varepsilon_1.$$

Therefore, again by the choice of ε_1 , we have

$$\|\alpha^j(g_1(a_0)) - g_1(a_j)\| < \varepsilon_0$$

for $0 \leq j \leq n-1$. Since $g_1(a_0)g_1(a_j) = 0$ for $1 \leq j \leq n-1$, it follows that

$$\|\alpha^j(g_1(a_0))g_1(a_0)\| < \varepsilon_0$$

for those j . Consequently, using $\|g_1(a_0)b - b\| < \varepsilon_0$ and $\|b \cdot g_1(a_0) - b\| < \varepsilon_0$, for $1 \leq j \leq n-1$ we get

$$\|\alpha^j(b)b\| < 4\varepsilon_0 + \|\alpha^j(b)\alpha^j(g_1(a_0))g_1(a_0)b\| < 5\varepsilon_0.$$

We note that

$$E' \cap pAp \cong \bigoplus_{l=1}^s p_{1,1}^{(l)} A p_{1,1}^{(l)},$$

which is a direct sum of hereditary subalgebras in A and therefore has real rank zero. So there exists a projection f in the hereditary subalgebra of $E' \cap pAp$ generated by $g_3(P(pa_0p))$ such that $\|fg_3(P(pa_0p)) - g_3(P(pa_0p))\| < \varepsilon_0$. Since $b \cdot g_3(P(pa_0p)) = g_3(P(pa_0p))$, we get $bf = fb = f$. Therefore $\|\alpha^j(f)f\| < 5\varepsilon_0$. Since $\alpha^n = \text{id}_A$, we get $\|\alpha^j(f)\alpha^k(f)\| < 5\varepsilon_0$ for $0 \leq j, k \leq n-1$ with $j \neq k$. By the choice $\varepsilon_0 \leq \frac{1}{5}\delta$ at the beginning of the proof, there are mutually orthogonal projections $e_0, e_1, \dots, e_{n-1} \in A$ such that $\|e_j - \alpha^j(f)\| < \frac{1}{10}\varepsilon$ for $0 \leq j \leq n-1$. We immediately get

$$\|\alpha(e_j) - e_{j+1}\| \leq \|\alpha(e_j) - \alpha^{j+1}(f)\| + \|\alpha^{j+1}(f) - e_{j+1}\| < \frac{2}{10}\varepsilon < \varepsilon$$

for $0 \leq j \leq n-2$. This is Condition (1) of Definition 1.1.

We now prove Condition (2). Let $a \in F$ and let $0 \leq j \leq n-1$. Because we assumed F is α -invariant, we have $\alpha^{-j}(a) \in F$. Choose $d_0 \in E$ such that $\|d_0 - pa\alpha^{-j}(a)p\| < \frac{1}{10}\varepsilon$. Set $d = d_0 + (1-p)\alpha^{-j}(a)(1-p)$. Since $\|a\| \leq 1$ we get $\|d\| < 1 + \frac{1}{10}\varepsilon$. Also

$$\begin{aligned} \|\alpha^{-j}(a) - d\| &\leq \|\alpha^{-j}(a) - [(1-p)\alpha^{-j}(a)(1-p) + p\alpha^{-j}(a)p]\| + \|p\alpha^{-j}(a)p - d_0\| \\ &\leq 2\|\alpha^{-j}(a)p - p\alpha^{-j}(a)\| + \|p\alpha^{-j}(a)p - d_0\| < \frac{2}{10}\varepsilon + \frac{1}{10}\varepsilon = \frac{3}{10}\varepsilon. \end{aligned}$$

By construction, $f \in E' \cap pAp$, so f commutes with d . From $\|f - \alpha^{-j}(e_j)\| < \frac{1}{10}\varepsilon$, we get

$$\|\alpha^{-j}(e_j)d - d\alpha^{-j}(e_j)\| \leq 2\|d\| \cdot \|f - \alpha^{-j}(e_j)\| < 2\left(1 + \frac{1}{10}\varepsilon\right)\frac{1}{10}\varepsilon.$$

Therefore, since $\varepsilon < 1$,

$$\begin{aligned} \|e_ja - ae_j\| &\leq 2\|\alpha^{-j}(a) - d\| + \|\alpha^{-j}(e_j)d - d\alpha^{-j}(e_j)\| \\ &< \frac{6}{10}\varepsilon + 2\left(1 + \frac{1}{10}\varepsilon\right)\frac{1}{10}\varepsilon \leq \frac{6}{10}\varepsilon + \frac{4}{10}\varepsilon = \varepsilon. \end{aligned}$$

This is Condition (2) of Definition 1.1.

It remains to verify Conditions (3) and (4) of Definition 1.1. This requires some work. Let X be the maximal ideal space of the unital C^* -algebra C generated by a_0, a_1, \dots, a_{n-1} , and let $h_0, h_1, \dots, h_{n-1}: X \rightarrow [0, 1]$ be the elements of $C(X)$ corresponding to a_0, a_1, \dots, a_{n-1} . Let μ be the probability measure on X such that if $h \in C(X)$ corresponds to an element $a \in C$, then $\int_X h \, d\mu = \tau(a)$. Then the functions h_j satisfy $h_j h_k = 0$ for $j \neq k$, and

$$0 \leq \sum_{j=0}^{n-1} h_j \leq 1 \quad \text{and} \quad \sum_{j=0}^{n-1} \int_X h_j \, d\mu > 1 - \min(\varepsilon_1, \frac{1}{16}\varepsilon_0^2).$$

Set

$$T_j = \{x \in X : h_j(x) \geq 1 - \frac{1}{4}\varepsilon_0\} \subset X.$$

For $x \notin \bigcup_{j=0}^{n-1} T_j$ we have $\sum_{j=0}^{n-1} h_j(x) < 1 - \frac{1}{4}\varepsilon_0$, whence

$$\begin{aligned} 1 - \frac{1}{16}\varepsilon_0^2 &\leq 1 - \min(\varepsilon_1, \frac{1}{16}\varepsilon_0^2) < \int_X \left(\sum_{j=0}^{n-1} h_j \right) d\mu \\ &\leq \sum_{j=0}^{n-1} \mu(T_j) + (1 - \frac{1}{4}\varepsilon_0) \left(1 - \sum_{j=0}^{n-1} \mu(T_j) \right) \\ &= 1 - \frac{\varepsilon_0}{4} \left(1 - \sum_{j=0}^{n-1} \mu(T_j) \right). \end{aligned}$$

It follows that

$$\sum_{j=0}^{n-1} \mu(T_j) > 1 - \left(\frac{4}{\varepsilon_0} \right) \left(\frac{\varepsilon_0^2}{16} \right) = 1 - \frac{1}{4}\varepsilon_0.$$

Since $g_3 \circ h_j \geq 1 - \frac{3}{4}\varepsilon_0$ on T_j , we get

$$\sum_{j=0}^{n-1} \tau(g_3(a_j)) = \sum_{j=0}^{n-1} \int_X (g_3 \circ h_j) d\mu \geq (1 - \frac{1}{4}\varepsilon_0) (1 - \frac{3}{4}\varepsilon_0) > 1 - \varepsilon_0.$$

By the choice of ε_1 , we have $\|g_3(a_j) - \alpha(g_3(a_{j+1}))\| < \varepsilon_0$ for $0 \leq j \leq n-2$. Since α preserves the trace, we get $\tau(g_3(a_{j+1})) < \tau(g_3(a_j)) + \varepsilon_0$, so inductively $\tau(g_3(a_j)) < \tau(g_3(a_0)) + j\varepsilon_0$. Therefore

$$\begin{aligned} 1 - \varepsilon_0 &< \sum_{j=0}^{n-1} \tau(g_3(a_j)) < \sum_{j=0}^{n-1} [\tau(g_3(a_0)) + j\varepsilon_0] \\ &< n\tau(g_3(a_0)) + \frac{1}{2}n(n-1)\varepsilon_0 \leq n\tau(g_3(a_0)) + (n^2 - 1)\varepsilon_0, \end{aligned}$$

that is,

$$\tau(g_3(a_0)) > \frac{1}{n} - n\varepsilon_0.$$

For the same reason as in a similar argument with g_2 earlier in the proof, we get

$$\|g_3(a_0) - [g_3((1-p)a_0(1-p)) + g_3(P(pa_0p))] \| < \varepsilon_0.$$

So, using $1-p \lesssim q_0$ at the last step,

$$\begin{aligned} \tau(g_3(P(pa_0p))) &> \tau(g_3(a_0)) - \tau(g_3((1-p)a_0(1-p))) - \varepsilon_0 \\ &> \frac{1}{n} - n\varepsilon_0 - \tau(1-p) - \varepsilon_0 > \frac{1}{n} - (n+1)\varepsilon_0 - \tau(q_0). \end{aligned}$$

Recalling that $\|fg_3(P(pa_0p)) - g_3(P(pa_0p))\| < \varepsilon_0$, we get

$$\|fg_3(P(pa_0p))f - g_3(P(pa_0p))\| < 2\varepsilon_0,$$

so that

$$\tau(f) \geq \tau(fg_3(P(pa_0p))f) > \tau(g_3(P(pa_0p))) - 2\varepsilon_0 > \frac{1}{n} - (n+3)\varepsilon_0 - \tau(q_0).$$

Because $\varepsilon < 1$, for each j the projection e_j is unitarily equivalent to $\alpha^j(f)$, whence $\tau(e_j) = \tau(\alpha^j(f)) = \tau(f)$. Combining this with

$$\tau(q_0) < \frac{\tau(q)}{2n} \quad \text{and} \quad \varepsilon_0 < \frac{\tau(q)}{2n(n+4)},$$

we get

$$\sum_{j=0}^{n-1} \tau(e_j) = n\tau(f) > 1 - n(n+3)\varepsilon_0 - n\tau(q_0) > 1 - \tau(q).$$

Therefore $e = \sum_{j=0}^{n-1} e_j$ satisfies $\tau(1 - e) < \tau(q)$. Since in simple C*-algebras with tracial rank zero, the traces determine the order on projections (Theorem 2.2), it follows that $1 - e \precsim q$. Since $q \in \overline{xAx}$, this proves Condition (3) of Definition 1.1.

If we use instead

$$\tau(q_0) < \frac{1}{4Nn^2} \quad \text{and} \quad \varepsilon_0 < \frac{1}{4Nn^2(n+4)},$$

we get instead $\tau(e) > 1 - \frac{1}{2Nn}$, or $\tau(1 - e) < \frac{1}{2Nn}$. Combining our estimate on $\tau(f)$ with

$$\tau(q_0) < \frac{1}{8n} \quad \text{and} \quad \varepsilon_0 < \frac{1}{8n(n+4)},$$

we get $\tau(f) > \frac{1}{2n}$. For every j , recalling from above that $\tau(e_j) = \tau(f)$, we have

$$\tau(e_j) = \tau(f) > \frac{1}{2n} > N\tau(1 - e).$$

Because the traces determine the order on projections, this implies that there are N mutually orthogonal projections in fAf , each Murray-von Neumann equivalent to $1 - e$. We have proved Condition (4) of Definition 1.1. ■

Theorem 8.2. Let A be an infinite dimensional simple unital C*-algebra with tracial rank zero and with a unique tracial state τ . Let $\alpha \in \text{Aut}(A)$ satisfy $\alpha^n = \text{id}_A$. Suppose that for every finite set $F \subset A$ and every $\varepsilon > 0$ there are positive elements $a_0, a_1, \dots, a_{n-1} \in A$ with $0 \leq a_j \leq 1$ such that:

- (1) $\|a_j a_k\| < \varepsilon$ for $j \neq k$.
- (2) $\|\alpha(a_j) - a_{j+1}\| < \varepsilon$ for $0 \leq j \leq n-2$.
- (3) $\|a_j c - c a_j\| < \varepsilon$ for $0 \leq j \leq n-1$ and all $c \in F$.
- (4) $\left| \tau \left(1 - \sum_{j=0}^{n-1} a_j \right) \right| < \varepsilon$.

Then the action of \mathbf{Z}_n generated by α has the tracial Rokhlin property.

Proof. We prove that the hypotheses imply those of Lemma 8.1. Let $F \subset A$ be finite and let $\varepsilon > 0$. Without loss of generality $\|c\| \leq 1$ for all $c \in F$. Let B be the universal (nonunital) C*-algebra generated by selfadjoint elements d_0, d_1, \dots, d_{n-1} subject to the relations $0 \leq d_j \leq 1$ and $d_j d_k = 0$ for $j \neq k$. Then B is isomorphic to the cone over \mathbf{C}^n , and is hence a projective C*-algebra. (See Lemmas 8.1.3 and 10.1.5 and Theorem 10.1.11 of [44].) Therefore it is semiprojective (Definition 14.1.3 of [44]). Thus, using Theorem 14.1.4 of [44] (see Definition 14.1.1 of [44], and take B there to be $\{0\}$), there is $\delta > 0$ such that whenever D is a C*-algebra and $d_0, d_1, \dots, d_{n-1} \in D$ are positive elements with $0 \leq d_j \leq 1$ and such that $\|d_j d_k\| < \delta$ for $j \neq k$, then there are positive elements $a_0, a_1, \dots, a_{n-1} \in D$ with $0 \leq a_j \leq 1$ such that $a_j a_k = 0$ for $j \neq k$ and

$$\|a_j - d_j\| < \min \left(\frac{1}{4}\varepsilon, \frac{1}{2}n^{-1}\varepsilon \right)$$

for all j . Apply our hypotheses with F as given and with $\min(\frac{1}{2}\varepsilon, \delta)$ in place of ε , obtaining $d_0, d_1, \dots, d_{n-1} \in B$, and let $a_0, a_1, \dots, a_{n-1} \in A$ be as above. The relation $a_j a_k = 0$ for $j \neq k$ is Condition (1) of Lemma 8.1. For (2), estimate

$$\|\alpha(a_j) - a_{j+1}\| \leq \|\alpha(d_j) - d_{j+1}\| + \|a_j - d_j\| + \|a_{j+1} - d_{j+1}\| < \frac{1}{2}\varepsilon + \frac{1}{4}\varepsilon + \frac{1}{4}\varepsilon = \varepsilon.$$

For (3), use $\|c\| \leq 1$ for $c \in F$ to estimate

$$\|a_j c - c a_j\| \leq \|d_j c - c d_j\| + 2\|a_j - d_j\| < \frac{1}{2}\varepsilon + 2 \left(\frac{1}{4}\varepsilon \right) = \varepsilon.$$

For (4), estimate

$$\tau \left(1 - \sum_{j=0}^{n-1} d_j \right) \leq \left| \tau \left(1 - \sum_{j=0}^{n-1} a_j \right) \right| + \sum_{j=0}^{n-1} \|a_j - d_j\| < \frac{1}{2}\varepsilon + n \left(\frac{1}{2}n^{-1}\varepsilon \right) = \varepsilon.$$

This completes the proof. \blacksquare

In the rest of this section, we will use the continuous field of rotation algebras A_θ , with section algebra A equal to the C^* -algebra of the discrete Heisenberg group, as described in Theorem 6.1, but we revert to tradition and use u_θ and v_θ for the unitary generators of A_θ and u, v , and z for the unitary generators of A . We let τ_θ be the standard trace on A_θ , as in Lemma 6.3. Since this continuous field will be the only one used in this section, we continue to follow Notation 6.2, letting $\Gamma(E)$ be the set of continuous sections of this field over the set E , and writing $a(\theta)$ for $a(\exp(2\pi i\theta))$ when a is a section.

Definition 8.3. Let A and A_θ be as in Theorem 6.1, with notation as above. The *noncommutative Fourier transform* on A_θ is the unique automorphism σ_θ of order 4 satisfying $\sigma_\theta(u_\theta) = v_\theta$ and $\sigma_\theta(v_\theta) = u_\theta^*$. The *noncommutative Fourier transform* on A is the unique automorphism σ of order 4 satisfying $\sigma(u) = v$, $\sigma(v) = u^*$, and $\sigma(z) = z$.

Lemma 8.4. Let the notation be as before and in Definition 8.3. Let $\varepsilon > 0$. Then there is $\delta > 0$ such that, with $I = \exp(2\pi i(-\delta, \delta))$, there are continuous sections $b_0, b_1, b_2, b_3 \in \Gamma(I)$ satisfying:

- (1) $0 \leq b_j(\theta) \leq 1$ for $0 \leq j \leq 3$ and $|\theta| < \delta$.
- (2) $\|b_j(\theta)b_k(\theta)\| < \varepsilon$ for $j \neq k$ and $|\theta| < \delta$.
- (3) Setting $b_4 = b_0$, we have $\|\sigma_\theta(b_j(\theta)) - b_{j+1}(\theta)\| < \varepsilon$ for $0 \leq j \leq 3$ and $|\theta| < \delta$.
- (4) $|\tau_\theta(1 - b_0(\theta) - b_1(\theta) - b_2(\theta) - b_3(\theta))| < \varepsilon$ for $|\theta| < \delta$.

Proof. Set $\varepsilon_0 = \frac{1}{16}\varepsilon$. Choose a continuous function $h: S^1 \rightarrow [0, 1]$ such that $h(\zeta) = 0$ for $\text{Im}(\zeta) \leq 0$ and $h(\zeta) = 1$ when $\zeta = \exp(2\pi it)$ with $t \in [\varepsilon_0, \frac{1}{2} - \varepsilon_0]$. Identifying A_0 with $C(S^1 \times S^1)$ in the obvious way, define elements $g_0, g_1, g_2, g_3 \in A_0$ by

$$\begin{aligned} g_0(\zeta_1, \zeta_2) &= h(\zeta_1)h(\zeta_2), & g_1(\zeta_1, \zeta_2) &= h(\bar{\zeta}_1)h(\zeta_2), \\ g_2(\zeta_1, \zeta_2) &= h(\bar{\zeta}_1)h(\bar{\zeta}_2), & \text{and} & & g_3(\zeta_1, \zeta_2) &= h(\zeta_1)h(\bar{\zeta}_2). \end{aligned}$$

Clearly $0 \leq g_j \leq 1$, $\sigma_0(g_j) = g_{j+1}$, and $g_j g_k = 0$ for $j \neq k$. Moreover,

$$\tau_0(g_0) \geq \left(\frac{1}{2} - 2\varepsilon_0 \right)^2 > \frac{1}{4} - 2\varepsilon_0,$$

and $\tau(g_j) = \tau(g_0)$ for all j , so

$$\tau_0(1 - g_0 - g_1 - g_2 - g_3) < 8\varepsilon_0 = \frac{1}{2}\varepsilon.$$

Since $\text{ev}_0: A \rightarrow A_0$ is surjective, there exist selfadjoint elements $a_0, a_1, a_2, a_3 \in A$ such that $\text{ev}_0(a_j) = g_j$ and $0 \leq a_j \leq 1$ for $0 \leq j \leq 3$. Since $\text{ev}_0(a_j)\text{ev}_0(a_k) = 0$ for $j \neq k$ and $\text{ev}_0(\sigma(a_j)) - \text{ev}_0(a_{j+1}) = 0$ for $0 \leq j \leq 2$, Theorem 6.1 provides $\delta_1 > 0$ such that whenever $|\theta| < \delta_1$, we have $\|\text{ev}_\theta(a_j)\text{ev}_\theta(a_k)\| < \varepsilon$ for $j \neq k$ and $\|\text{ev}_\theta(\sigma(a_j)) - \text{ev}_\theta(a_{j+1})\| < \varepsilon$ for $0 \leq j \leq 3$. Since

$$|\tau_0(1 - \text{ev}_0(a_0) - \text{ev}_0(a_1) - \text{ev}_0(a_2) - \text{ev}_0(a_3))| < \frac{1}{2}\varepsilon,$$

Lemma 6.3 provides $\delta_2 > 0$ such that

$$|\tau_\theta(1 - \text{ev}_\theta(a_0) - \text{ev}_\theta(a_1) - \text{ev}_\theta(a_2) - \text{ev}_\theta(a_3))| < \frac{1}{2}\varepsilon$$

for $|\theta| < \delta_2$. The lemma is now proved by taking $\delta = \min(\delta_1, \delta_2)$ and taking b_j to be the restriction of a_j , regarded as a section over S^1 of the continuous field of Theorem 6.1, to $I = \exp(2\pi i(-\delta, \delta))$. ■

Lemma 8.5. Let $\theta \in \mathbf{R} \setminus \mathbf{Q}$. Then for every $\varepsilon > 0$ there is $n \in \mathbf{N}$ with $n > 0$ such that

$$\text{dist}(n\theta, \mathbf{Z}) < \varepsilon \quad \text{and} \quad \text{dist}(n^2\theta, \mathbf{Z}) < \varepsilon.$$

Proof. Define $h: S^1 \times S^1 \rightarrow S^1 \times S^1$ by

$$h(\zeta_1, \zeta_2) = (\exp(2\pi i\theta)\zeta_1, \exp(2\pi i\theta)\zeta_1^2\zeta_2).$$

As in the discussion preceding Proposition 1.5 of [24], the map h is a minimal homeomorphism of $S^1 \times S^1$, and the forward orbit of $(1, 1)$ consists of all points $(\exp(2\pi i n\theta), \exp(2\pi i n^2\theta))$ for $n \in \mathbf{N}$. The result is immediate from the density of the forward orbit. ■

Proposition 8.6. Let $\theta \in \mathbf{R} \setminus \mathbf{Q}$. Then the action of \mathbf{Z}_4 on A_θ generated by the noncommutative Fourier transform σ_θ of Definition 8.3 has the tracial Rokhlin property.

Proof. Let the notation be as before and in Definition 8.3. It suffices to verify the conditions of Theorem 8.2. Moreover, we may take the finite set F to be $F = \{u, v\}$. We know from Theorem 7.6 that A_θ has tracial rank zero, and we know that A_θ has a unique tracial state τ_θ .

Accordingly, let $\varepsilon > 0$. Apply Lemma 8.4, obtaining $\delta_0 > 0$, an arc $I = \exp(2\pi i(-\delta_0, \delta_0))$, and continuous sections $b_0, b_1, b_2, b_3 \in \Gamma(I)$ satisfying Properties (1)–(4) there. Choose polynomials f_0, f_1, f_2, f_3 in four noncommuting variables such that in $A_0 = C(S^1 \times S^1)$ we have

$$\|f_j(u_0, u_0^*, v_0, v_0^*) - b_j(0)\| < \frac{1}{5}\varepsilon.$$

for $0 \leq j \leq 3$. Because we are dealing with a continuous field (by Theorem 6.1), there is $\delta_1 > 0$ with $\delta_1 \leq \delta_0$ such that for $|\eta| < \delta_1$ and $0 \leq j \leq 3$ we have

$$\|f_j(u_\eta, u_\eta^*, v_\eta, v_\eta^*) - b_j(\eta)\| < \frac{2}{5}\varepsilon.$$

Choose (see Proposition 4.3 of [5]) $\delta_2 > 0$ such that whenever D is a C^* -algebra and $x_1, x_2, x_3, x_4, y \in D$ are elements of norm 1 which satisfy $\|[x_k, y]\| < \delta_2$ for all k , then

$$\|[f_j(x_1, x_2, x_3, x_4), y]\| < \frac{1}{5}\varepsilon$$

for $0 \leq j \leq 3$. Now use Lemma 8.5 to choose $n \in \mathbf{N}$ with $n > 0$ and $m \in \mathbf{Z}$ such that

$$|\exp(2\pi i n\theta) - 1| < \delta_2 \quad \text{and} \quad |n^2\theta - m| < \delta_1.$$

Let $\eta = n^2\theta - m$, and let $\psi: A_\eta \rightarrow A_\theta$ be the homomorphism determined by $\psi(u_\eta) = u_\theta^n$ and $\psi(v_\eta) = v_\theta^m$. Set $a_j = \psi(b_j(\eta))$ for $0 \leq j \leq 3$. We verify the hypotheses (1)–(4) of Theorem 8.2. We have $0 \leq a_j \leq 1$ because $0 \leq b_j(\eta) \leq 1$. For (1), we have $\|a_j a_k\| < \varepsilon$ for $j \neq k$ because $|\eta| < \delta_1 \leq \delta_0$ implies $\|b_j(\eta) b_k(\eta)\| < \varepsilon$ for $j \neq k$. For (2), we observe that $\sigma_\theta \circ \psi = \psi \circ \sigma_\eta$. Since $|\eta| < \delta_0$ we have $\|\sigma_\eta(b_j(\eta)) - b_{j+1}(\eta)\| < \varepsilon$, and this implies $\|\sigma_\theta(a_j) - a_{j+1}\| < \varepsilon$. For (4), we note that η and θ are both irrational, so uniqueness of the traces on A_θ and A_η implies $\tau_\theta \circ \psi = \tau_\eta$. Therefore $|\eta| < \delta_0$ implies

$$|\tau_\theta(1 - a_0 - a_1 - a_2 - a_3)| = |\tau_\eta(1 - b_0(\eta) - b_1(\eta) - b_2(\eta) - b_3(\eta))| < \varepsilon.$$

Finally, we prove (3). We have $u_\theta^n v_\theta = \exp(-2\pi i n\theta) v_\theta u_\theta^n$, so

$$\|u_\theta^n v_\theta - v_\theta u_\theta^n\| = |\exp(-2\pi i n\theta) - 1| < \delta_2.$$

For either similar reasons or trivially, we get $\|xy - yx\| < \delta_2$ for all

$$x \in \{\psi(u_\eta), \psi(u_\eta)^*, \psi(v_\eta), \psi(v_\eta)^*\} \quad \text{and} \quad y \in \{u_\theta, v_\theta\}.$$

Therefore, by the choice of δ_2 ,

$$\|[\psi(f_j(u_\eta, u_\eta^*, v_\eta, v_\eta^*)), y]\| < \frac{1}{5}\varepsilon$$

for $y \in \{u_\theta, v_\theta\}$. Furthermore, from $|\eta| < \delta_1$ we get

$$\|\psi(f_j(u_\eta, u_\eta^*, v_\eta, v_\eta^*)) - a_j\| < \frac{2}{5}\varepsilon.$$

Therefore

$$\begin{aligned} \| [a_j, u_\theta] \| &\leq \| [\psi(f_j(u_\eta, u_\eta^*, v_\eta, v_\eta^*)), u_\theta] \| + 2\| \psi(f_j(u_\eta, u_\eta^*, v_\eta, v_\eta^*)) - a_j \| \\ &< \frac{1}{5}\varepsilon + \frac{4}{5}\varepsilon = \varepsilon, \end{aligned}$$

and similarly $\| [a_j, v_\theta] \| < \varepsilon$. This shows that $\| [a_j, y] \| < \varepsilon$ for $y \in F$, and completes the proof of (3).

Having verified the hypotheses of Theorem 8.2, we conclude from that theorem that the action of \mathbf{Z}_4 generated by σ_θ has the tracial Rokhlin property. ■

Next, we show that the crossed product by the noncommutative Fourier transform satisfies the Universal Coefficient Theorem. We formulate this condition in a more convenient way than the usual definition. This reformulation will be more important in Section 10 than here.

Definition 8.7. Let A be a separable nuclear C*-algebra. We say that A *satisfies the Universal Coefficient Theorem* if for every separable C*-algebra B such that $K_*(B)$ is an injective abelian group, the natural map

$$\gamma = \gamma_{A,B} : KK^*(A, B) \rightarrow \text{Hom}(K_*(A), K_*(B))$$

is an isomorphism.

Remark 8.8. The definition in [57] is that A satisfies the Universal Coefficient Theorem if for every separable C*-algebra B (actually, every C*-algebra B with a countable approximate identity; see Remark 8.9 below), there is a natural short exact sequence

$$0 \longrightarrow \text{Ext}_1^{\mathbf{Z}}(K_*(A), K_*(B)) \longrightarrow KK^*(A, B) \longrightarrow \text{Hom}(K_*(A), K_*(B)) \longrightarrow 0,$$

in which:

- (1) The second map is the map γ of Definition 8.7.
- (2) The first map has degree one and is the inverse of the map

$$\text{Ker}(\gamma) \rightarrow \text{Ext}_1^{\mathbf{Z}}(K_*(A), K_*(B))$$

which sends a class in $KK^1(A, B)$ represented by an extension

$$0 \longrightarrow A \longrightarrow E \longrightarrow B \longrightarrow 0$$

to the Ext class of the short exact sequence

$$0 \longrightarrow K_*(A) \longrightarrow K_*(E) \longrightarrow K_*(B) \longrightarrow 0,$$

and is the suspension of this on $KK^0(A, B)$.

This statement is unwieldy (both in the definition and verification) if the maps are defined, and imprecise if they are not. Moreover, the condition in Definition 8.7 is what one in practice verifies. The proof of Theorem 4.1 of [57] shows that if A is nuclear and satisfies the condition of Definition 8.7, then for every separable C^* -algebra B the map $\text{Ker}(\gamma) \rightarrow \text{Ext}_1^{\mathbf{Z}}(K_*(A), K_*(B))$ of (2) above is in fact invertible, and the resulting sequence

$$0 \longrightarrow \text{Ext}_1^{\mathbf{Z}}(K_*(A), K_*(B)) \longrightarrow KK^*(A, B) \longrightarrow \text{Hom}(K_*(A), K_*(B)) \longrightarrow 0,$$

is in fact exact.

Remark 8.9. The blanket assumption in [57] on the second variable B is that it has a countable approximate identity. There seems to be a gap in the proof of Theorem 4.1 of [57] in this generality, because it is not clear that the algebra D_0 occurring in the proof has a countable approximate identity.

Much of the following proof is due to Walters. See Theorem 9.4 of [61].

Lemma 8.10. Let $\theta \in \mathbf{R} \setminus \mathbf{Q}$. Then the crossed product $C^*(\mathbf{Z}_4, A_\theta, \sigma_\theta)$, with σ_θ as in Definition 8.3, satisfies the Universal Coefficient Theorem.

Proof. Let A , A_θ , and $\text{ev}_\theta : A \rightarrow A_\theta$ be as in Theorem 6.1, and let $\sigma \in \text{Aut}(A)$ be as in Definition 8.3. Then the crossed product $C^*(\mathbf{Z}_4, A, \sigma)$ is the C^* -algebra of the semidirect product of \mathbf{Z}_4 and the discrete Heisenberg group H , formed from the obvious corresponding order four automorphism of H . This group is amenable. Using Lemma 3.5 of [59], we may apply Proposition 10.7 of [59], specialized to groups, to H , and conclude that $C^*(\mathbf{Z}_4, A, \sigma)$ satisfies the Universal Coefficient Theorem. If $\theta \in \mathbf{Q}$, then A_θ is type I, so $C^*(\mathbf{Z}_4, A_\theta, \sigma_\theta)$ is type I by Theorem 4.1 of [49]. Therefore it satisfies the Universal Coefficient Theorem.

Now let $\theta_1, \theta_2 \in [0, 1)$ satisfy $\theta_1 < \theta_2$. Let $I_{\theta_1, \theta_2}^{(0)}$ be the set of sections of the continuous field of Theorem 6.1 which vanish on the closed arc from $\exp(2\pi i\theta_1)$ to $\exp(2\pi i\theta_2)$, and let $J_{\theta_1, \theta_2}^{(0)}$ be the set of sections which vanish on the other closed arc with the same endpoints. Let I_{θ_1, θ_2} and J_{θ_1, θ_2} be the crossed products of these by the action of \mathbf{Z}_4 generated by σ . Using the maps ev_{θ_1} and ev_{θ_2} , we obtain a \mathbf{Z}_4 -equivariant short exact sequence of C^* -algebras

$$0 \longrightarrow I_{\theta_1, \theta_2}^{(0)} \oplus J_{\theta_1, \theta_2}^{(0)} \longrightarrow A \longrightarrow A_{\theta_1} \oplus A_{\theta_2} \longrightarrow 0.$$

Since crossed products preserve exact sequences (see Lemma 2.8.2 of [46]), we obtain the short exact sequence

$$\begin{aligned} 0 \longrightarrow I_{\theta_1, \theta_2} \oplus J_{\theta_1, \theta_2} \longrightarrow C^*(\mathbf{Z}_4, A, \sigma) \\ \longrightarrow C^*(\mathbf{Z}_4, A_{\theta_1}, \sigma_{\theta_1}) \oplus C^*(\mathbf{Z}_4, A_{\theta_2}, \sigma_{\theta_2}) \longrightarrow 0. \end{aligned}$$

For $\theta_1, \theta_2 \in \mathbf{Q}$ we have seen that the second and third terms of this sequence satisfy the Universal Coefficient Theorem, so $I_{\theta_1, \theta_2} \oplus J_{\theta_1, \theta_2}$ satisfies the Universal Coefficient Theorem by Proposition 2.3(a) of [57]. It is easy to check from Definition 8.7 that I_{θ_1, θ_2} and J_{θ_1, θ_2} therefore each separately satisfy the Universal Coefficient Theorem.

Now let $\theta \in (0, 1)$ be arbitrary. Choose

$$\beta_1, \beta_2, \dots, \gamma_1, \gamma_2, \dots \in (0, 1) \cap \mathbf{Q}$$

such that

$$\beta_1 < \beta_2 < \dots < \theta < \dots < \gamma_2 < \gamma_1 \quad \text{and} \quad \lim_{n \rightarrow \infty} \beta_n = \lim_{n \rightarrow \infty} \gamma_n = \theta.$$

Let L be the kernel of the map of crossed products

$$C^*(\mathbf{Z}_4, A, \sigma) \rightarrow C^*(\mathbf{Z}_4, A_\theta, \sigma_\theta)$$

Then, since crossed products preserve direct limits and exact sequences,

$$J_{\beta_1, \gamma_1} \subset J_{\beta_2, \gamma_2} \subset \cdots \quad \text{and} \quad L = \overline{\bigcup_{n=1}^{\infty} J_{\beta_n, \gamma_n}}.$$

So L satisfies the Universal Coefficient Theorem by Proposition 2.3(b) of [57], whence $C^*(\mathbf{Z}_4, A_\theta, \sigma_\theta)$ satisfies the Universal Coefficient Theorem by Proposition 2.3(a) of [57]. ■

Theorem 8.11. Let $\theta \in \mathbf{R} \setminus \mathbf{Q}$. Then the crossed product $C^*(\mathbf{Z}_4, A_\theta, \sigma_\theta)$, with σ_θ as in Definition 8.3, is a simple AH algebra with slow dimension growth and real rank zero.

Proof. We know from Theorem 7.6 that A_θ has tracial rank zero. Combining Proposition 8.6 and Theorem 2.7, we find that the crossed product $C^*(\mathbf{Z}_4, A_\theta, \sigma_\theta)$ has tracial rank zero. By Lemma 8.10 it satisfies the Universal Coefficient Theorem, by Corollary 1.12 it is simple, and it is clearly separable and nuclear. The result therefore follows from Lemma 7.5. ■

The following corollary is Theorem 9.3 of [61].

Corollary 8.12. There exists a dense G_δ -set $E \subset \mathbf{R} \setminus \mathbf{Q}$ such that for every $\theta \in E$, the crossed product $C^*(\mathbf{Z}_4, A_\theta, \sigma_\theta)$ is a simple AF algebra.

Proof. It is shown in [60] that there is a dense G_δ -set $E \subset \mathbf{R} \setminus \mathbf{Q}$ such that for every $\theta \in E$, the crossed product $C^*(\mathbf{Z}_4, A_\theta, \sigma_\theta)$ has trivial K_1 and torsion-free K_0 . The result therefore follows from Theorem 8.11 and Lemma 7.5. ■

Corollary 8.13. There exists a dense G_δ -set $E \subset \mathbf{R} \setminus \mathbf{Q}$ such that for every $\theta \in E$, the fixed point algebra $A_\theta^{\sigma_\theta}$ of the automorphism $\sigma_\theta \in \text{Aut}(A_\theta)$ is a simple AF algebra.

Proof. For every θ in the dense G_δ -set $E \subset \mathbf{R} \setminus \mathbf{Q}$ of Corollary 8.12, Proposition 4.5 shows that $A_\theta^{\sigma_\theta}$ is strongly Morita equivalent to a simple AF algebra. For these θ , it follows that $A_\theta^{\sigma_\theta}$ is a simple AF algebra. ■

9. OTHER FINITE CYCLIC GROUP ACTIONS ON THE IRRATIONAL ROTATION ALGEBRA

In this section, we consider the crossed products of the irrational rotation algebra by the automorphisms of orders 3 and 6 coming from the action of $\text{SL}_2(\mathbf{Z})$. There are four differences from the previous section. First, the actions don't extend to an action on the C^* -algebra of the discrete Heisenberg group, and we must use in its place a slightly larger group and corresponding larger continuous field. Second, the analysis of the case $\theta = 0$, in the proof of the analog of Lemma 8.4, is messier. Third, rather than doing everything twice we prove a lemma which reduces the tracial Rokhlin property for the action of \mathbf{Z}_3 to the case of \mathbf{Z}_6 . Finally, lacking information analogous to that of [60] on the K-theory, we do not prove that any of the crossed products are AF.

As in Section 8, for $\theta \in \mathbf{R}$ let A_θ be the ordinary rotation algebra, the universal C^* -algebra generated by unitaries u_θ and v_θ satisfying $v_\theta u_\theta = \exp(2\pi i \theta) u_\theta v_\theta$.

Definition 9.1. Define an automorphism $\varphi_\theta: A_\theta \rightarrow A_\theta$ of order 6 by

$$\varphi_\theta(u_\theta) = v_\theta \quad \text{and} \quad \varphi_\theta(v_\theta) = e^{-\pi i \theta} u_\theta^* v_\theta.$$

One readily checks that φ_θ really does define an automorphism of order 6. This is the automorphism of order 6 discussed in the introduction to Section 8.

Remark 9.2. A computation easily shows that φ_θ^2 is the automorphism of A_θ order 3 discussed in the introduction to Section 8.

The following result is the analog of Theorem 6.1. We need the slight modification here because φ_θ does not come from an automorphism of the discrete Heisenberg group, as a result of the factor $\exp(-\pi i \theta)$ in the definition of $\varphi(v_\theta)$.

Lemma 9.3. Let H be the group on generators x, y , and z , subject to the relations

$$yx = z^2 xy, \quad zx = xz, \quad \text{and} \quad zy = yz.$$

Let $B = C^*(H)$. Then there is a continuous field of C^* -algebras over S^1 whose fiber over $\exp(2\pi i \theta)$ is $A_{2\theta}$, whose C^* -algebra of continuous sections is B , and such that the evaluation map $\text{ev}_\theta: B \rightarrow A_{2\theta}$ of sections at $\exp(2\pi i \theta)$ is determined by

$$\text{ev}_\theta(x) = u_{2\theta}, \quad \text{ev}_\theta(y) = v_{2\theta}, \quad \text{and} \quad \text{ev}_\theta(z) = \exp(4\pi i \theta) \cdot 1.$$

Moreover, there is a unique automorphism φ of B of order 6 such that

$$\varphi(x) = y, \quad \varphi(y) = z^* x^* y, \quad \text{and} \quad \varphi(z) = z,$$

and $\text{ev}_\theta \circ \varphi = \varphi_{2\theta} \circ \text{ev}_\theta$ for all θ .

Proof. Apply Corollary 3.6 of [53], with the group being \mathbf{Z} , to the continuous field over S^1 whose fiber over $\exp(2\pi i \theta)$ is $C(S^1)$ for all θ , with standard generator $u_{2\theta}$, and with the action on that fiber being generated by the automorphism $\alpha_\theta(u_{2\theta}) = \exp(4\pi i \theta) u_{2\theta}$. The section algebra of this continuous field is $C(S^1 \times S^1)$, generated by two sections: the constant section x whose value at $\exp(2\pi i \theta)$ is the standard generator of $C(S^1)$ (called $u_{2\theta}$ above), and the section $z(\exp(2\pi i \theta)) = \exp(2\pi i \theta) \cdot 1$ for all θ . It is the universal C^* -algebra on unitary generators x and z satisfying $zx = xz$. The action of \mathbf{Z} on the sections is generated by the automorphism $\alpha(z) = z$ and $\alpha(x) = z^2 x$. Therefore the crossed product is $C^*(H)$, with y being the implementing unitary for the action. Corollary 3.6 of [53] thus shows that there is a Hilbert-continuous field with the desired properties, and the remark in Definition 3.3 of [53] shows that there is a continuous field.

One checks that the formula for φ defines a unique homomorphism by checking the commutation relations for the images of the generators. A computation shows that $\varphi^6 = \text{id}_B$, so φ is an automorphism. It is immediate to check that $\text{ev}_\theta \circ \varphi = \varphi_{2\theta} \circ \text{ev}_\theta$. ■

Remark 9.4. By comparing the actions, one immediately sees that a section b of the continuous field of Lemma 9.3 over a small neighborhood of 0 is continuous if and only if the section $\zeta \mapsto b(\zeta^2)$ is a continuous section of the continuous field of Theorem 6.1. (In fact, the continuous field of Lemma 9.3 is really just the pullback of that of Theorem 6.1 by the map $\zeta \mapsto \zeta^2$.)

Lemma 9.5. Let $\varepsilon > 0$. Then there is $\delta > 0$ such that there are continuous sections b_0, b_1, \dots, b_5 of the continuous field of Lemma 9.3 defined on the arc $I = \exp(2\pi i(-\delta, \delta))$ and satisfying:

- (1) $0 \leq b_j(\theta) \leq 1$ for $0 \leq j \leq 5$ and $|\theta| < \delta$.

- (2) $\|b_j(\theta)b_k(\theta)\| < \varepsilon$ for $j \neq k$ and $|\theta| < \delta$.
- (3) Setting $b_6 = b_0$, we have $\|\varphi_\theta(b_j(\theta)) - b_{j+1}(\theta)\| < \varepsilon$ for $0 \leq j \leq 5$ and $|\theta| < \delta$.
- (4) With τ_θ being the standard trace on A_θ as in Lemma 6.3, we have the estimate $\tau_{2\theta} \left(1 - \sum_{j=0}^5 b_j(\theta) \right) < \varepsilon$ for $|\theta| < \delta$.

Proof. Set $\varepsilon_0 = \frac{1}{12}\varepsilon$.

Define $h_0: \mathbf{R}^2 \rightarrow \mathbf{R}^2$ by $h_0(r_1, r_2) = (r_1 - r_2, r_1)$. Then h_0 is an invertible linear map whose matrix is in $\mathrm{GL}_2(\mathbf{Z})$, so $h_0(\mathbf{Z}^2) = \mathbf{Z}^2$ and h_0 induces a homeomorphism $h: \mathbf{R}^2/\mathbf{Z}^2 \rightarrow \mathbf{R}^2/\mathbf{Z}^2$. With the obvious identification of A_0 with $C(\mathbf{R}^2/\mathbf{Z}^2)$, one checks that $\varphi_0(f) = f \circ h^{-1}$ for all $f \in C(\mathbf{R}^2/\mathbf{Z}^2)$.

Define open sets $E_0, E_1, \dots, E_5 \subset \mathbf{R}^2$ as follows. Let $\mathrm{Conv}(S)$ denote the convex hull of a subset $S \subset \mathbf{R}^2$. Then set

$$\begin{aligned} E_0 &= \mathrm{int} (\mathrm{Conv} (\{(0,0), (1,0), (\frac{2}{3}, \frac{1}{3})\})) , \\ E_1 &= \mathrm{int} (\mathrm{Conv} (\{(0,0), (1,1), (\frac{1}{3}, \frac{2}{3})\})) , \\ E_2 &= \mathrm{int} (\mathrm{Conv} (\{(1,0), (1,1), (\frac{2}{3}, \frac{1}{3})\})) , \\ E_3 &= \mathrm{int} (\mathrm{Conv} (\{(0,1), (1,1), (\frac{1}{3}, \frac{2}{3})\})) , \\ E_4 &= \mathrm{int} (\mathrm{Conv} (\{(0,0), (1,1), (\frac{2}{3}, \frac{1}{3})\})) , \\ E_5 &= \mathrm{int} (\mathrm{Conv} (\{(0,0), (0,1), (\frac{1}{3}, \frac{1}{2})\})) . \end{aligned}$$

These sets can be described as follows. Divide $[0,1]^2$ in half along the main diagonal $r_1 = r_2$. Divide each of the two resulting triangles using line segments from the point $(\frac{1}{3}, \frac{2}{3})$ or $(\frac{2}{3}, \frac{1}{3})$ as appropriate to the three vertices. The E_j for even j are in the lower right triangle.

These 6 sets are disjoint open subsets of $[0,1]^2$ whose union is dense in $[0,1]^2$ and each of which has measure $\frac{1}{6}$. Calculations show that

$$\begin{aligned} h_0(E_0) &= E_1, \quad h_0(E_1) = E_2 + (-1, 0), \quad h_0(E_2) = E_3, \\ h_0(E_3) &= E_4 + (-1, 0), \quad h_0(E_4) = E_5, \quad \text{and} \quad h_0(E_5) = E_0 + (-1, 0). \end{aligned}$$

Since $(-1, 0) \in \mathbf{Z}^2$, the images U_j of the E_j in $\mathbf{R}^2/\mathbf{Z}^2$ are disjoint open sets with measure $\frac{1}{6}$ which are cyclically permuted by h .

Choose a closed set $T_0 \subset U_0$ with measure greater than $\frac{1}{6} - \varepsilon_0$, and choose $f_0 \in C(\mathbf{R}^2/\mathbf{Z}^2)$ with $0 \leq g \leq 1$, with $\mathrm{supp}(g) \subset U_0$, and such that $g_0(x) = 1$ for all $x \in T_0$. Define $g_j = g_0 \circ h^{-j}$ for $1 \leq j \leq 5$. Since the sets U_j are pairwise disjoint, we have $g_j g_k = 0$ for $j \neq k$. Moreover,

$$\tau_0 \left(1 - \sum_{j=0}^5 g_j \right) < 1 - 6\varepsilon_0.$$

The proof is now finished as in the last paragraph of the proof of Lemma 8.4. We use Lemma 9.3 in place of Theorem 6.1. Lemma 6.3 still implies that traces of continuous sections are continuous, by Remark 9.4. ■

Proposition 9.6. Let $\theta \in \mathbf{R} \setminus \mathbf{Q}$. Then the action of \mathbf{Z}_6 on A_θ generated by the automorphism φ_θ of Definition 9.1 has the tracial Rokhlin property.

Proof. Except for one point (see below), the proof is essentially the same as for Proposition 8.6. We use Lemma 9.3 in place of Theorem 6.1. We use Lemma 9.5 in place of Lemma 8.4, adjusting for the fact that $B_\theta \cong A_{2\theta}$ rather than A_θ .

The difference is in the proof that $\psi \circ \varphi_\eta = \varphi_\theta \circ \psi$. With δ_1 and δ_2 chosen to satisfy conditions analogous to those in the proof of Proposition 8.6, we require $n \in \mathbf{N}$ with $n > 0$ and $m \in \mathbf{Z}$ such that

$$|\exp(2\pi in\theta) - 1| < \delta_2 \quad \text{and} \quad |n^2\theta - m| < \delta_1,$$

and in addition such that m is even. To get this, apply Lemma 8.5 with $\frac{1}{2}\theta$ in place of θ , obtaining $n \in \mathbf{N}$ with $n > 0$ and $m_0 \in \mathbf{Z}$ such that

$$|\exp(\pi in\theta) - 1| < \frac{1}{2}\delta_2 \quad \text{and} \quad \left| \frac{1}{2}n^2\theta - m_0 \right| < \frac{1}{2}\delta_1.$$

Then take $m = 2m_0$.

This done, as in the proof of Proposition 8.6, set $\eta = n^2\theta - m$ and let $\psi: A_\eta \rightarrow A_\theta$ be the homomorphism such that $\psi(u_\eta) = u_\theta^n$ and $\psi(v_\eta) = v_\theta^n$. We must check that $\psi \circ \varphi_\eta = \varphi_\theta \circ \psi$. First, use the relation $v_\theta u_\theta^* = \exp(-2\pi i\theta)u_\theta^*v_\theta$ to get $u_\theta^{-n}v_\theta^n = \exp(\pi in(n-1)\theta)(u_\theta^*v_\theta)^n$. Further, since m is even, we get $\exp(-\pi i\eta) = \exp(-\pi in^2\theta)$. Now

$$\psi \circ \varphi_\eta(u_\eta) = \psi(v_\eta) = v_\theta^n = \varphi_\theta(u_\theta^n) = \varphi_\theta \circ \psi(u_\eta)$$

and

$$\begin{aligned} \psi \circ \varphi_\eta(v_\eta) &= \psi(e^{-\pi i\eta}u_\eta^*v_\eta) = e^{-\pi i\eta}u_\theta^{-n}v_\theta^n = e^{-\pi in^2\theta}e^{\pi in(n-1)\theta}(u_\theta^*v_\theta)^n \\ &= (e^{-\pi i\theta}u_\theta^*v_\theta)^n = \varphi_\theta(v_\theta)^n = \varphi_\theta \circ \psi(v_\eta). \end{aligned}$$

This completes the proof. ■

Rather than repeating everything for the action of \mathbf{Z}_3 , we prove the following lemma.

Lemma 9.7. Let A be a stably finite simple unital C*-algebra, and let $\alpha \in \text{Aut}(A)$ generate an action of \mathbf{Z}_n which has the tracial Rokhlin property. Let $n = lm$ be a nontrivial factorization in positive integers. Then α^l generates an action of \mathbf{Z}_m which has the tracial Rokhlin property.

Proof. Let $F \subset A$ be a finite set, let $\varepsilon > 0$, let $N \in \mathbf{N}$, and let $x \in A$ be a nonzero positive element. Apply Lemma 1.10 with ε/l^2 in place of ε , and with the given values of F , N , and x . Call the resulting projections f_0, f_1, \dots, f_{n-1} . For $0 \leq k \leq m-1$, define

$$e_k = \sum_{j=kl}^{kl+l-1} f_j.$$

Then

$$\begin{aligned} \|\alpha^l(e_k) - e_{k+1}\| &\leq \sum_{j=kl}^{kl+l-1} \|\alpha^l(f_j) - f_{j+l}\| \\ &\leq \sum_{j=kl}^{kl+l-1} \sum_{r=0}^{l-1} \|\alpha(f_{j+r}) - f_{j+r+1}\| < l^2(\varepsilon/l^2) = \varepsilon. \end{aligned}$$

For $a \in F$, we have

$$\|ae_k - e_k a\| \leq \sum_{j=kl}^{kl+l-1} \|af_j - f_j a\| < l(\varepsilon/l^2) < \varepsilon.$$

This verifies Conditions (1) and (2) of Definition 1.1. Conditions (3) and (4) are immediate since $\sum_{k=0}^{l-1} e_k = \sum_{j=0}^{n-1} f_j$, and since every e_k dominates some f_j . ■

Corollary 9.8. Let $\theta \in \mathbf{R} \setminus \mathbf{Q}$, and let φ_θ be as in Definition 9.1. Then the action of \mathbf{Z}_3 on A_θ generated by φ_θ^2 has the tracial Rokhlin property.

Proof. Combine Proposition 9.6 and Lemma 9.7. ■

Lemma 9.9. Let $\theta \in \mathbf{R} \setminus \mathbf{Q}$, and let φ_θ be as in Definition 9.1. Then the crossed products $C^*(\mathbf{Z}_6, A_\theta, \varphi_\theta)$ and $C^*(\mathbf{Z}_3, A_\theta, \varphi_\theta^2)$ satisfy the Universal Coefficient Theorem.

Proof. The proof is the same as for Lemma 8.10, once we know that the group H in Lemma 9.3 is amenable. This is immediate from the fact that the subgroup generated by x , y , and z^2 is an amenable subgroup (in fact, the discrete Heisenberg group) of index 2. ■

Theorem 9.10. Let $\theta \in \mathbf{R} \setminus \mathbf{Q}$, and let φ_θ be as in Definition 9.1. Then the crossed products $C^*(\mathbf{Z}_6, A_\theta, \varphi_\theta)$ and $C^*(\mathbf{Z}_3, A_\theta, \varphi_\theta^2)$ are simple AH algebras with slow dimension growth and real rank zero.

Proof. The proof is the same as for Theorem 8.11. ■

We don't know whether any of these algebras are AF, because we don't know the K-theory. However, the K-theory computations for θ rational, done in [21] and [20], suggest that all should be AF. These computations are for the fixed point algebras rather than the crossed products, but one expects the K-theory for the irrational fixed point algebras to be the same as for the generic rational ones, and in the irrational case Proposition 9.6, Corollary 9.8, Corollary 1.12, and Proposition 4.5 show that the fixed point algebra is always Morita equivalent to the crossed product.

10. THE CROSSED PRODUCT OF A SIMPLE NONCOMMUTATIVE TORUS BY THE FLIP

In this section, we prove that the crossed product of any higher dimensional simple noncommutative torus by the flip is a simple AF algebra. This generalizes Theorem 3.1 of [6], and completely answers a question raised in the introduction to [19]. As in Section 8, there are three parts: the proof that the action has the tracial Rokhlin property, the proof that the crossed product satisfies the Universal Coefficient Theorem, and the computation of the K-theory of the crossed product.

Definition 10.1. Let θ be a skew symmetric real $d \times d$ matrix, and let A_θ be the corresponding noncommutative torus with unitary generators

$$u_1(\theta), u_2(\theta), \dots, u_d(\theta)$$

being the unitaries u_1, u_2, \dots, u_d of Notation 5.1. When no confusion is likely to result, we will suppress θ in the notation. Then the *flip automorphism* $\sigma_\theta: A_\theta \rightarrow A_\theta$ is the automorphism of order two determined by $\sigma_\theta(u_k) = u_k^*$ for $1 \leq k \leq n$. We further set, in this section, $B_\theta = C^*(\mathbf{Z}_2, A_\theta, \sigma_\theta)$, and call it the crossed product by the flip.

Notation 10.2. For $d \geq 1$ let R_d be the set of all skew symmetric real $d \times d$ matrices θ such that $\theta_{j,k} \in [-1, 1]$ for all j and k . For any d , we let $r: R_{d+1} \rightarrow R_d$ be the restriction map $r(\theta) = \theta|_{\mathbf{Z}^d \times \{0\}}$, which deletes the last row and column of θ . (See Remark 5.4. The intended value of d will always be clear.) For any continuous field C over R_{d-1} , we let $r^*(C)$ denote the continuous field over R_d obtained as the pullback of C . (See [11], especially Sections 10.1 and 10.3, for information on continuous fields of C^* -algebras. See Lemma 1.3 of [32] for pullbacks of continuous fields.) In a variation of Notation 6.2, we denote the set of all continuous sections of C by $\Gamma(C)$.

Lemma 10.3. Let the notation be as in Notation 10.2.

- (1) The space R_d is homeomorphic to $[-1, 1]^{d(d-1)/2}$. In particular, $R_1 = \{0\}$.
- (2) For any continuous field C over R_d , there is an isomorphism $\varphi: \Gamma(r^*(C)) \rightarrow C([0, 1]^d, \Gamma(C))$ such that every evaluation map $\text{ev}_\theta: \Gamma(r^*(C)) \rightarrow C_{r(\theta)} = r^*(C)_\theta$ factors as

$$\Gamma(r^*(C)) \xrightarrow{\varphi} C([0, 1]^d, \Gamma(C)) \xrightarrow{\text{ev}_t} \Gamma(C) \xrightarrow{\text{ev}_{r(\theta)}} C_{r(\theta)}$$

for some $t \in [0, 1]^d$.

- (3) For $\theta \in R_d$ and a continuous field C over R_d , the map $\text{ev}_\theta: \Gamma(r^*(C)) \rightarrow C_{r(\theta)}$ is the composition of $\text{ev}_{r(\theta)}: \Gamma(C) \rightarrow C_{r(\theta)}$ and a homotopy equivalence.

Proof. Part (1) is immediate. For Part (2), we use the homeomorphism from R_{d+1} to $[0, 1]^d \times R_d$ given by

$$\rho \mapsto (\rho_{d+1,1}, \rho_{d+1,2}, \dots, \rho_{d+1,d}, r(\rho))$$

to construct φ . Part (3) then follows from the fact that the homomorphism ev_t in Part (2) is a homotopy equivalence. ■

Lemma 10.4. Let the notation be as in Definition 10.1 and Notation 10.2.

- (1) For every skew symmetric real $d \times d$ matrix θ there is $\theta' \in R_d$ such that $A_{\theta'} \cong A_\theta$ and $B_{\theta'} \cong B_\theta$.
- (2) There exists a continuous field $A^{(d)}$ of C^* -algebras over R_d , whose fiber over $\theta \in R_d$ is A_θ , and such that the sections

$$\theta \mapsto f(\theta)u_1(\theta)^{n_1}u_2(\theta)^{n_2} \cdots u_d(\theta)^{n_d},$$

for $f \in C(R_d)$ and $n_1, n_2, \dots, n_d \in \mathbf{Z}$, span a dense subalgebra of the C^* -algebra of all continuous sections.

- (3) There is an automorphism σ of the C^* -algebra $\Gamma(A^{(d)})$ of continuous sections of the continuous field of Part (3) such that for any section a , we have $\sigma(a)(\theta) = \sigma_\theta(a(\theta))$ for all θ . Moreover, there exists a continuous field $B^{(d)}$ of C^* -algebras over R_d , whose fiber over $\theta \in R_d$ is B_θ , and whose C^* -algebra of continuous sections is $C^*(\mathbf{Z}_2, \Gamma(A^{(d)}), \sigma)$.

Proof. Part (1) is immediate from the fact that A_θ is unchanged if some entry $\theta_{j,k}$ is replaced by $\theta_{j,k} + n$ for some $n \in \mathbf{Z}$. Part (2) is contained in the discussion following Corollary 2.9 of [53], but we will need to reprove it anyway in the course of proving Part (3).

To prove Part (3), we prove by induction on d that both $A^{(d)}$ and $B^{(d)}$ are Hilbert continuous fields in the sense of Definition 3.3 of [53]. The comment in the

last part of that definition shows that such a field is automatically continuous in the usual sense, so this does in fact prove Part (3) (as well as Part (2)).

If $d = 1$ then R_d consists of just one point, and the statement is trivial. Suppose now that the result is known for d . Since $A^{(d)}$ is a Hilbert continuous field, it is immediate from Lemma 10.3(2) that $r^*(A^{(d)})$ is as well. Let \mathbf{Z} act on the fiber $r^*(A^{(d)})_\theta = A_{r(\theta)}$ according to the automorphism of Lemma 5.5, so that the crossed product is A_θ . From the formula given in the proof, it is clear that we get a continuous field of actions of \mathbf{Z} in the sense of Definition 3.1 of [53]. So Theorem 3.5 of [53] shows that $A^{(d+1)}$ is Hilbert continuous and has the right algebra of sections. Moreover, the actions σ_θ define a continuous field of actions of \mathbf{Z}_2 , so the same theorem now yields Hilbert continuity of $B^{(d+1)}$. ■

We now prove that the flip has the tracial Rokhlin property. We use the method of proof of Proposition 6.14. There is one difficulty: the restriction of the flip to the irrational rotation subalgebra $A_\eta \subset A_\theta$ that we construct need not be the flip on that algebra, but rather has the form

$$v_\eta \mapsto \lambda v_\eta^* \quad \text{and} \quad w_\eta \mapsto \zeta w_\eta^*$$

for some a priori unknown $\lambda, \zeta \in S^1$. Fortunately, all such automorphisms are conjugate to the flip, and the set of possibilities is compact.

Proposition 10.5. Let $\theta \in R_d$ be nondegenerate (Definition 5.6). Then the action of \mathbf{Z}_2 on A_θ generated by the automorphism σ_θ of Definition 10.1 has the tracial Rokhlin property.

Proof. We verify the conditions of Theorem 8.2. Theorem 7.6 implies that A_θ has tracial rank zero, and Theorem 5.9 implies that A_θ has a unique tracial state.

For the rest of the proof, let the notation for rotation algebras be as in Theorem 6.1, Notation 6.2, and Lemma 6.3. In particular, A is the C*-algebra of the discrete Heisenberg group.

Let $\varepsilon > 0$ and let $F \subset A_\theta$ be a finite set. Choose and fix $\eta_0 \in \mathbf{R} \setminus \mathbf{Q}$.

For $\eta \in \mathbf{R}$ and $\lambda, \zeta \in S^1$ let $\sigma_\eta^{(\lambda, \zeta)} \in \text{Aut}(A_\eta)$ be the automorphism of order two determined by

$$\sigma_\eta^{(\lambda, \zeta)}(v_\eta) = \lambda v_\eta^* \quad \text{and} \quad \sigma_\eta^{(\lambda, \zeta)}(w_\eta) = \zeta w_\eta^*.$$

Thus $\sigma_\eta^{(1,1)} = \sigma_\eta$. Moreover, we claim that $\sigma_\eta^{(\lambda, \zeta)}$ is conjugate to σ_η . To see this, choose $\lambda_0, \zeta_0 \in S^1$ such that $\lambda_0^2 = \lambda$ and $\zeta_0^2 = \zeta$, let $\varphi \in \text{Aut}(A_\eta)$ be the automorphism determined by

$$\varphi(v_\eta) = \lambda_0 v_\eta \quad \text{and} \quad \varphi(w_\eta) = \zeta_0 w_\eta,$$

and check that $\sigma_\eta \circ \varphi = \varphi \circ \sigma_\eta^{(\lambda, \zeta)}$.

It follows from Proposition 8.6 and Lemma 9.7 that σ_η has the tracial Rokhlin property for $\eta \notin \mathbf{Q}$, so that each $\sigma_\eta^{(\lambda, \zeta)}$ also has the tracial Rokhlin property. Taking $\eta = \eta_0$, for each $\lambda, \zeta \in S^1$ choose orthogonal projections $p_0^{(\lambda, \zeta)}, p_1^{(\lambda, \zeta)} \in A_{\eta_0}$ such that

$$\|\sigma_{\eta_0}^{(\lambda, \zeta)}(p_0^{(\lambda, \zeta)}) - p_1^{(\lambda, \zeta)}\| < \frac{1}{2}\varepsilon \quad \text{and} \quad \tau_{\eta_0}(1 - p_0^{(\lambda, \zeta)} - p_1^{(\lambda, \zeta)}) < \frac{1}{2}\varepsilon.$$

Choose continuous selfadjoint sections $b_0^{(\lambda, \zeta)}, b_1^{(\lambda, \zeta)} \in A$ with $0 \leq b_j^{(\lambda, \zeta)} \leq 1$ and such that $b_j^{(\lambda, \zeta)}(\eta_0) = p_j^{(\lambda, \zeta)}$.

We claim that for fixed $\lambda_0, \zeta_0 \in S^1$ the function on $S^1 \times S^1 \times \mathbf{R}$ defined by

$$(\lambda, \zeta, \eta) \mapsto \|\sigma_\eta^{(\lambda, \zeta)}(b_0^{(\lambda_0, \zeta_0)}(\eta)) - b_1^{(\lambda_0, \zeta_0)}(\eta)\|$$

is jointly continuous. Identify the continuous function algebra $C(S^1 \times S^1, A)$ with the universal C*-algebra generated by unitaries v, w, x, y, z subject to the relations that $wv = zvw$ and x, y, z are all central, by taking $A = C^*(v, w, z)$ and identifying x and y with the functions $(\lambda, \zeta) \mapsto \lambda$ and $(\lambda, \zeta) \mapsto \zeta$. Using Theorem 6.1, this algebra can be realized in an obvious way as the section algebra of a continuous field over $S^1 \times S^1 \times S^1$ whose fiber over $(\lambda, \zeta, \exp(2\pi i\eta))$ is A_η . Let $\sigma \in \text{Aut}(A)$ be determined by

$$\sigma(v) = xv^*, \quad \sigma(w) = yw^*, \quad \sigma(x) = x, \quad \sigma(y) = y, \quad \text{and} \quad \sigma(z) = z.$$

The evaluation map $\text{ev}_{\lambda, \zeta, \exp(2\pi i\eta)}: C(S^1 \times S^1, A) \rightarrow A_\eta$ satisfies

$$\sigma_\eta^{(\lambda, \zeta)} \circ \text{ev}_{\lambda, \zeta, \exp(2\pi i\eta)} = \text{ev}_{\lambda, \zeta, \exp(2\pi i\eta)} \circ \sigma,$$

and since σ sends continuous sections to continuous sections the claim follows.

Using the claim, for every $\lambda_0, \zeta_0 \in S^1$ there are open neighborhoods U of (λ_0, ζ_0) and V of η_0 such that

$$\|\sigma_\eta^{(\lambda, \zeta)}(b_0^{(\lambda_0, \zeta_0)}(\eta)) - b_1^{(\lambda_0, \zeta_0)}(\eta)\| < \varepsilon$$

for $(\lambda, \zeta) \in U$ and $\eta \in V$. Using compactness of $S^1 \times S^1$ to cover it with finitely many of the sets U , and intersecting the corresponding sets V , we find a neighborhood W_0 of η_0 and finitely many pairs of sections chosen from the ones above, which for simplicity we call $b_0^{(m)}, b_1^{(m)}$ for $1 \leq m \leq r$, such that for every $(\lambda, \zeta) \in S^1$ there is m such that for every $\eta \in W_0$ we have

$$\|\sigma_\eta^{(\lambda, \zeta)}(b_0^{(m)}(\eta)) - b_1^{(m)}(\eta)\| < \varepsilon.$$

Since $b_0^{(m)}(\eta_0)b_1^{(m)}(\eta_0) = 0$ and $\tau_{\eta_0}(1 - b_0^{(m)}(\eta_0) - b_1^{(m)}(\eta_0)) < \frac{1}{2}\varepsilon$, we can use the continuous field structure and Lemma 6.3 to find an open set W with $\eta_0 \in W \subset W_0$ such that for every $(\lambda, \zeta) \in S^1 \times S^1$ there is m such that for every $\eta \in W$ we have:

- (1) $0 \leq b_0^{(m)}(\eta), b_1^{(m)}(\eta) \leq 1$.
- (2) $\|b_0^{(m)}(\eta)b_1^{(m)}(\eta)\| < \varepsilon$.
- (3) $\|\sigma_\eta^{(\lambda, \zeta)}(b_0^{(m)}(\eta)) - b_1^{(m)}(\eta)\| < \varepsilon$.
- (4) $|\tau_\eta(1 - b_0^{(m)}(\eta) - b_1^{(m)}(\eta))| < \varepsilon$.

Apply Lemma 6.13 with θ, η_0, F , and ε as given, with $k = n = 1$, with $U = \exp(2\pi iW)$, and with $S = \{b_0^{(m)}|_U, b_1^{(m)}|_U : 1 \leq m \leq r\}$. We obtain $\eta \in \mathbf{R} \setminus \mathbf{Q}$ and

$$l = (l_1, l_2, \dots, l_d) \in \mathbf{Z}^d \quad \text{and} \quad m = (m_1, m_2, \dots, m_d) \in \mathbf{Z}^d.$$

Set

$$x = u_1^{l_1}u_2^{l_2} \cdots u_d^{l_d} \quad \text{and} \quad y = u_1^{m_1}u_2^{m_2} \cdots u_d^{m_d},$$

so that $yx = \exp(2\pi i\eta)xy$. Let $\varphi: A_\eta \rightarrow A_\theta$ be the homomorphism such that $\varphi(v_\eta) = x$ and $\varphi(w_\eta) = y$. We get

$$\|[\varphi(b_j^{(m)}(\eta)), c]\| < \varepsilon$$

for all $c \in F$, for $1 \leq m \leq r$, and for $j = 0, 1$.

Using the commutation relations in Notation 5.1, there are $\lambda, \zeta \in S^1$ such that

$$x^* = \overline{\lambda}(u_1^*)^{l_1}(u_2^*)^{l_2} \cdots (u_d^*)^{l_d} \quad \text{and} \quad y^* = \overline{\zeta}(u_1^*)^{m_1}(u_2^*)^{m_2} \cdots (u_d^*)^{m_d}.$$

It follows that $\sigma_\theta \circ \varphi = \varphi \circ \sigma_\eta^{(\lambda, \zeta)}$. By the above, there exists m such that conditions (1)–(4) above hold with this λ, ζ, η . Set $a_j = \varphi(b_j^{(m)}(\eta))$ for $j = 0, 1$. We have $0 \leq a_0, a_1 \leq 1$, and we claim that a_0 and a_1 satisfy the hypotheses (1)–(4) of Theorem 8.2. Parts (1) and (4) follow from Conditions (2) and (4) above, Part (2) follows from Condition (3) above and the relation $\sigma_\theta \circ \varphi = \varphi \circ \sigma_\eta^{(\lambda, \zeta)}$, and Part (3) follows from the commutator estimate at the end of the previous paragraph.

It now follows from Theorem 8.2 that the action of \mathbf{Z}_2 on A_θ generated by σ_θ has the tracial Rokhlin property. ■

Now we turn to K-theory. In [36] and [19], the K-theory calculations used the exact sequence of [45]. However, for the Universal Coefficient Theorem we need information about KK-theory, so we use [47].

Notation 10.6. Throughout this section, we let G be the infinite dihedral group $G = \mathbf{Z} \rtimes \mathbf{Z}_2$. We let g_0 be the generator of \mathbf{Z} corresponding to 1, and we let h_0 be the nontrivial element of \mathbf{Z}_2 , so that the defining relations are $h_0^2 = 1$ and $h_0 g_0 h_0^{-1} = g_0^{-1}$. We further let $H_0 = \{1, h_0\}$ be the obvious copy of \mathbf{Z}_2 . Note that also $h_1 = g_0 h_0$ satisfies $h_1^2 = 1$, and set $H_1 = \{1, h_1\}$, another subgroup isomorphic to \mathbf{Z}_2 . It is known that G is the free product of H_0 and H_1 ; see for example the discussion of the structure of this group in the preliminaries in [36].

Lemma 10.7. With notation as in Notation 10.6 and Notation 10.2, there is a continuous field α of actions of G on $r^*(A^{(d-1)})$ (in the sense of Definition 3.1 of [52]), determined by actions $\alpha^{(\theta)}$ of G on $r^*(A^{(d-1)})_\theta$, such that:

- (1) $\alpha_{g_0}^{(\theta)}$ acts on the generators of $r^*(A^{(d-1)})_\theta = A_{r(\theta)}$ by the formula $\alpha_{g_0}^{(\theta)}(u_j) = \exp(2\pi i \theta_{j,d}) u_j$ for $1 \leq j \leq d-1$, and the continuous field of crossed products $\theta \mapsto C^*(\mathbf{Z}, r^*(A^{(d-1)})_\theta, \alpha_{g_0}^{(\theta)})$ is $A^{(d)}$.
- (2) $\alpha_{h_0}^{(\theta)}$ acts on the generators of $r^*(A^{(d-1)})_\theta = A_{r(\theta)}$ by $\alpha_{g_0}^{(\theta)}(u_j) = u_j^*$ for $1 \leq j \leq d-1$, and the continuous field $\theta \mapsto C^*(\mathbf{Z}_2, r^*(A^{(d-1)})_\theta, \alpha_{h_0}^{(\theta)})$ of crossed products is $r^*(B^{(d-1)})$ with $B^{(d-1)}$ as in Lemma 10.4(3).
- (3) α_{h_1} is conjugate to α_{h_0} via the continuous family of automorphisms $\theta \mapsto \varphi^{(\theta)}$ determined by $\varphi^{(\theta)}(u_j) = \exp(-\pi i \alpha_{j,d}) u_j$ for $1 \leq j \leq d-1$.
- (4) The continuous field of crossed products $\theta \mapsto C^*(G, r^*(A^{(d-1)})_\theta, \alpha^{(\theta)})$ is $B^{(d)}$.

Proof. Lemma 5.5 shows that there exist automorphisms $\beta^{(\theta)}$ on the individual fibers with the properties claimed for $\alpha_{g_0}^{(\theta)}$ in Part (1). Let S be the set of all continuous sections of $\Gamma(r^*(A^{(d-1)}))$ of the form

$$\theta \mapsto f(\theta) u_1(\theta)^{n_1} u_2(\theta)^{n_2} \cdots u_{d-1}(\theta)^{n_{d-1}}$$

with $f \in C(R_d)$. It is easy to check that if $s \in S$ then $\theta \mapsto \beta^{(\theta)}(s(\theta))$ is again in S , hence continuous. Since $\text{Span}(S)$ is dense in $\Gamma(r^*(A^{(d-1)}))$, this implies that $\theta \mapsto \beta^{(\theta)}$ defines an automorphism of $r^*(A^{(d-1)})$.

Applying the pullback via r to the automorphism of Lemma 10.4(3), we obtain an automorphism $\theta \mapsto \gamma^{(\theta)}$ of $r^*(A^{(d-1)})$ with the properties claimed in Part (2). A computation shows that

$$(\gamma^{(\theta)})^2 = \text{id}_{r^*(A^{(d-1)})_\theta} \quad \text{and} \quad \gamma^{(\theta)} \circ \beta^{(\theta)} \circ (\gamma^{(\theta)})^{-1} = (\beta^{(\theta)})^{-1}$$

for all θ , from which it follows that there is an action of G satisfying (1) and (2).

To get Part (4), we rewrite

$$\begin{aligned} C^*(G, r^*(A^{(d-1)})_\theta, \alpha^{(\theta)}) &= C^*(\mathbf{Z} \rtimes \mathbf{Z}_2, r^*(A^{(d-1)})_\theta, \alpha^{(\theta)}) \\ &= C^*(\mathbf{Z}_2, C^*(\mathbf{Z}, r^*(A^{(d-1)})_\theta, \alpha_{g_0}^{(\theta)}), \bar{\gamma}^{(\theta)}), \end{aligned}$$

where $\bar{\gamma}^{(\theta)}$ acts on $r^*(A^{(d-1)})_\theta$ via $\alpha_{h_0}^{(\theta)}$ and on the canonical unitary u in the crossed product by \mathbf{Z} like conjugation by h_0 on g_0 , that is, $\bar{\gamma}^{(\theta)}(u) = u^{-1}$. Identifying $u_j(r(\theta))$ with $u_j(\theta)$ and u with $u_d(\theta)$, we identify $\bar{\gamma}^{(\theta)}$ with σ_θ as in Lemma 10.4(3).

It remains to prove Part (3). A computation shows that

$$\varphi^{(\theta)} \circ \alpha_{g_0 h_0}^{(\theta)} = \alpha_{h_0}^{(\theta)} \circ \varphi^{(\theta)}$$

for all θ . ■

Lemma 10.8. Let $d \in \mathbf{N}$ and let $\theta \in R_d$. Then for every separable C^* -algebra D there is a commutative diagram with exact columns:

$$\begin{array}{ccc} & \downarrow & \downarrow \\ KK^{1-j}(D, \Gamma(B^{(d)})) & \xrightarrow{(\text{ev}_\theta)_*} & KK^{1-j}(D, B_\theta) \\ \downarrow & & \downarrow \\ KK^j(D, \Gamma(r^*(A^{(d-1)}))) & \xrightarrow{(\text{ev}_\theta)_*} & KK^j(D, A_{r(\theta)}) \\ \downarrow & & \downarrow \\ KK^j(D, \Gamma(r^*(B^{(d-1)}))) \\ \oplus KK^j(D, \Gamma(r^*(B^{(d-1)}))) & \xrightarrow{(\text{ev}_\theta)_*} & KK^j(D, B_{r(\theta)}) \\ \oplus KK^j(D, B_{r(\theta)}) \\ \downarrow & & \downarrow \\ KK^j(D, \Gamma(B^{(d)})) & \xrightarrow{(\text{ev}_\theta)_*} & KK^j(D, B_\theta) \\ \downarrow & & \downarrow \\ KK^{1-j}(D, \Gamma(r^*(A^{(d-1)}))) & \xrightarrow{(\text{ev}_\theta)_*} & KK^{1-j}(D, A_{r(\theta)}) \\ \downarrow & & \downarrow \end{array}$$

Moreover, for every separable nuclear C*-algebra D there is a commutative diagram with exact columns:

$$\begin{array}{ccc}
 & \downarrow & \\
 KK^{1-j}(A_{r(\theta)}, D) & \xrightarrow{(\text{ev}_\theta)^*} & KK^{1-j}(\Gamma(r^*(A^{(d-1)})), D) \\
 \downarrow & & \downarrow \\
 KK^j(B_\theta, D) & \xrightarrow{(\text{ev}_\theta)^*} & KK^j(\Gamma(B^{(d)}), D) \\
 \downarrow & & \downarrow \\
 KK^j(B_{r(\theta)}, D) & \xrightarrow{(\text{ev}_\theta)^*} & KK^j(\Gamma(r^*(B^{(d-1)})), D) \\
 \oplus KK^j(B_{r(\theta)}, D) & & \oplus KK^j(\Gamma(r^*(B^{(d-1)})), D) \\
 \downarrow & & \downarrow \\
 KK^j(A_{r(\theta)}, D) & \xrightarrow{(\text{ev}_\theta)^*} & KK^j(\Gamma(r^*(A^{(d-1)})), D) \\
 \downarrow & & \downarrow \\
 KK^{1-j}(B_\theta, D) & \xrightarrow{(\text{ev}_\theta)^*} & KK^{1-j}(\Gamma(B^{(d)}), D) \\
 \downarrow & & \downarrow
 \end{array}$$

Proof. We derive these exact sequences from [47], so we begin by constructing a suitable action of G on a suitable tree X . We take the vertices to be $X^0 = \mathbf{Z}$, and the oriented edges to be

$$X^1 = \{(2n - 1, 2n), (2n + 1, 2n) : n \in \mathbf{Z}\}.$$

We define the actions of g_0 and h_0 to be

$$g_0 \cdot n = n + 2 \quad \text{and} \quad h_0 \cdot n = -n$$

on X^0 , and

$$g_0 \cdot (2n - 1, 2n) = (2n + 1, 2n + 2), \quad g_0 \cdot (2n + 1, 2n) = (2n + 3, 2n + 2)$$

and

$$h_0 \cdot (2n - 1, 2n) = (-2n + 1, -2n), \quad h_0 \cdot (2n + 1, 2n) = (-2n - 1, -2n)$$

on X^1 . One proves that these define an action by checking that the maps on X^0 and X^1 are bijective and satisfy the two defining relations of Notation 10.6. Following the discussion after Lemma 3 of [47], we choose subsets $\Sigma^0 \subset X^0$ and $\Sigma^1 \subset X^1$ which are representatives of the quotients of X^0 and X^1 by G . We choose $\Sigma^0 = \{0, 1\}$ and $\Sigma^1 = \{(1, 0)\}$. The stabilizers are then

$$G_0 = \{1, h_0\}, \quad G_1 = \{1, g_0 h_0\}, \quad \text{and} \quad G_{(1,0)} = \{1\}.$$

We now apply the exact sequence of Theorem 16 of [47] for the full crossed products, to the actions of G on both $\Gamma(r^*(A^{(d-1)}))$ and on $A_{r(\theta)}$. By naturality we obtain a commutative diagram with exact columns, in which for simplicity we suppress the

notation for the actions in the crossed products.

$$\begin{array}{ccc}
& \downarrow & \downarrow \\
KK^{1-j}(D, C^*(G, \Gamma(r^*(A^{(d-1)})))) & \xrightarrow{(\text{ev}_\theta)_*} & KK^{1-j}(D, C^*(G, A_{r(\theta)})) \\
\downarrow & & \downarrow \\
KK^j(D, \Gamma(r^*(A^{(d-1)}))) & \xrightarrow{(\text{ev}_\theta)_*} & KK^j(D, A_{r(\theta)}) \\
\downarrow & & \downarrow \\
KK^j(D, C^*(G_0, \Gamma(r^*(A^{(d-1)})))) & \xrightarrow{(\text{ev}_\theta)_*} & KK^j(D, C^*(G_0, A_{r(\theta)})) \\
\oplus KK^j(D, C^*(G_1, \Gamma(r^*(A^{(d-1)})))) & & \oplus KK^j(D, C^*(G_1, A_{r(\theta)})) \\
\downarrow & & \downarrow \\
KK^j(D, C^*(G, \Gamma(r^*(A^{(d-1)})))) & \xrightarrow{(\text{ev}_\theta)_*} & KK^j(D, C^*(G, A_{r(\theta)})) \\
\downarrow & & \downarrow \\
KK^{1-j}(D, \Gamma(r^*(A^{(d-1)}))) & \xrightarrow{(\text{ev}_\theta)_*} & KK^{1-j}(D, A_{r(\theta)}) \\
\downarrow & & \downarrow
\end{array}$$

Then we use Lemma 10.7 to identify $C^*(G, \Gamma(r^*(A^{(d-1)}))) \cong \Gamma(B^{(d)})$, etc. Note that Lemma 10.7(3) ensures that the crossed products by G_1 are all isomorphic to the corresponding crossed products by G_0 . This gives the first diagram of the present lemma.

To get the second one, we use Theorem 17 of [47] in place of Theorem 16 of [47], and proceed the same way. ■

Proposition 10.9. For every skew symmetric real $d \times d$ matrix θ , we have $K_0(B_\theta) \cong \mathbf{Z}^{3 \cdot 2^{d-1}}$ and $K_1(B_\theta) = 0$.

Proof. By Lemma 10.4(1), it suffices to consider $\theta \in R_d$. We prove by induction on d that the maps

$$(\text{ev}_\theta)_*: K_*(\Gamma(A^{(d)})) \rightarrow K_*(A_\theta) \quad \text{and} \quad (\text{ev}_\theta)_*: K_*(\Gamma(B^{(d)})) \rightarrow K_*(B_\theta)$$

are isomorphisms for all θ . The proof of Theorem 7 of [19] shows that $K_*(B_\theta)$ is as claimed in the statement for at least some values of $\theta \in R_d$, so it will follow that this is correct for all $\theta \in R_d$.

When $d = 1$, the space R_d consists of a single point. So the only possible ev_θ is an isomorphism of C^* -algebras. Suppose the result is known for all $\theta \in R_d$ for some d . Let $\theta \in R_{d+1}$. In the first diagram in Lemma 10.8, replace d by $d+1$, and use this θ . Also take $D = \mathbf{C}$, giving a diagram in ordinary K-theory. The induction hypothesis and Lemma 10.3(3) imply that

$$\text{ev}_\theta: \Gamma(r^*(A^{(d)})) \rightarrow A_{r(\theta)} \quad \text{and} \quad \text{ev}_\theta: \Gamma(r^*(B^{(d)})) \rightarrow B_{r(\theta)}$$

are isomorphisms on K-theory. Now $(\text{ev}_\theta)_*: K_*(\Gamma(B^{(d+1)})) \rightarrow K_*(B_\theta)$ is an isomorphism by the Five Lemma. That $(\text{ev}_\theta)_*: K_*(\Gamma(A^{(d+1)})) \rightarrow K_*(A_\theta)$ is an isomorphism follows in the same way from a similar diagram in which the columns are

obtained from the Pimsner-Voiculescu exact sequence for the K-theory of crossed products by \mathbf{Z} [48]. This completes the induction, and the proof. ■

Proposition 10.10. For every skew symmetric real $d \times d$ matrix θ , the algebra B_θ satisfies the Universal Coefficient Theorem. (See Definition 8.7 and Remark 8.8.)

Proof. By Lemma 10.4(1), it suffices to consider $\theta \in R_d$. We prove this by induction on d . When $d = 1$, the only possible algebra is $B_0 = C^*(\mathbf{Z}_2, C(S^1), \sigma_0)$, which satisfies the Universal Coefficient Theorem because it is type I.

Suppose the result is known for all $\theta \in R_d$ for some d . Let $\theta \in R_{d+1}$. Let D be any separable nuclear C*-algebra such that $K_*(D)$ is an injective abelian group. Using the diagrams of Lemma 10.8, we can construct the commutative diagram:

$$\begin{array}{ccc}
 & \downarrow & \downarrow \\
 KK^{1-j}(A_{r(\theta)}, D) & \xrightarrow{(\text{ev}_\theta)^*} & \text{Hom}(K_*(A_{r(\theta)}), K_*(D))_{1-j} \\
 \downarrow & & \downarrow \\
 KK^j(B_\theta, D) & \xrightarrow{(\text{ev}_\theta)^*} & \text{Hom}(K_*(B_\theta), K_*(D))_j \\
 \downarrow & & \downarrow \\
 KK^j(B_{r(\theta)}, D) & \xrightarrow{(\text{ev}_\theta)^*} & \text{Hom}(K_*(B_{r(\theta)}), K_*(D))_j \\
 \oplus KK^j(B_{r(\theta)}, D) & & \oplus \text{Hom}(K_*(B_{r(\theta)}), K_*(D))_j \\
 \downarrow & & \downarrow \\
 KK^j(A_{r(\theta)}, D) & \xrightarrow{(\text{ev}_\theta)^*} & \text{Hom}(K_*(A_{r(\theta)}), K_*(D))_j \\
 \downarrow & & \downarrow \\
 KK^{1-j}(B_\theta, D) & \xrightarrow{(\text{ev}_\theta)^*} & \text{Hom}(K_*(B_\theta), K_*(D))_{1-j} \\
 \downarrow & & \downarrow
 \end{array}$$

The left column is exact by Lemma 10.8, and the right column is exact because $K_*(D)$ is injective and by Lemma 10.8. Using Theorem 1.17 of [57] (see the preceding discussion for the definition of \mathcal{N}), it follows from Theorem 7.6 that $A_{r(\theta)}$ satisfies the Universal Coefficient Theorem. Also, $B_{r(\theta)}$ satisfies the Universal Coefficient Theorem by the induction hypothesis. So B_θ satisfies the Universal Coefficient Theorem by the Five Lemma. ■

Theorem 10.11. Let θ be a nondegenerate skew symmetric real $d \times d$ matrix, with $d \geq 2$. Then $C^*(\mathbf{Z}_2, A_\theta, \sigma_\theta)$ is a simple AF algebra.

Proof. Combining Proposition 10.5 and Theorem 2.7, we find that the crossed product $B_\theta = C^*(\mathbf{Z}_2, A_\theta, \sigma_\theta)$ has tracial rank zero. By Proposition 10.10 it satisfies the Universal Coefficient Theorem, and it is clearly separable and nuclear. Proposition 10.9 shows that $K_0(B_\theta)$ is torsion free and $K_1(B_\theta) = 0$. The result therefore follows from Lemma 7.5. ■

11. MORE ON THE ROKHLIN PROPERTY AND TRACIAL APPROXIMATE INNERNESS

In this section we prove, in analogy with Theorem 2.7, that the crossed product of an AF algebra by an action with the strict Rokhlin property is again an AF algebra. We also prove that tracially approximately inner automorphisms act trivially on the trace space and on K_0 mod infinitesimals. These results are relevant to the examples we construct in the next section.

It is useful to start the first proof with the following analog of Lemma 2.3.

Lemma 11.1. Let A be a unital C^* -algebra, and let $\alpha \in \text{Aut}(A)$ be an automorphism which satisfies $\alpha^n = \text{id}_A$ and such that the action of \mathbf{Z}_n generated by α has the strict Rokhlin property (Definition 1.3). Then for every finite set $F \subset A$, every finite dimensional subalgebra $E \subset A$, and every $\varepsilon > 0$, there are mutually orthogonal projections $e_0, e_1, \dots, e_{n-1} \in A$ and a unitary $v \in A$ such that:

- (1) $\|\alpha(e_j) - e_{j+1}\| < \varepsilon$ for $0 \leq j \leq n-1$, where, following Convention 1.2, we take $e_n = e_0$.
- (2) $\|e_j a - a e_j\| < \varepsilon$ for $0 \leq j \leq n-1$ and all $a \in F$.
- (3) $\sum_{j=0}^{n-1} e_j = 1$.
- (4) $\|v - 1\| < \varepsilon$, and e_j commutes with all elements of vEv^* for $0 \leq j \leq n-1$.

Proof. The proof is a slightly simpler version of the proof of Lemma 2.3. ■

Theorem 11.2. Let A be a unital AF algebra, and let $\alpha \in \text{Aut}(A)$ be an automorphism which satisfies $\alpha^n = \text{id}_A$ and such that the action of \mathbf{Z}_n generated by α has the strict Rokhlin property. Then $C^*(\mathbf{Z}_n, A, \alpha)$ is an AF algebra.

Proof. We prove that for every finite set $S \subset C^*(\mathbf{Z}_n, A, \alpha)$ and every $\varepsilon > 0$, there is a finite dimensional C^* -subalgebra $D \subset C^*(\mathbf{Z}_n, A, \alpha)$ such that every element of S is within ε of an element of D . Theorem 2.2 of [7] will then imply that $C^*(\mathbf{Z}_n, A, \alpha)$ is AF. Let $u \in C^*(\mathbf{Z}_n, A, \alpha)$ be the canonical unitary implementing the automorphism α . Without loss of generality we may assume that $S = F \cup \{u\}$ for a finite subset F of the unit ball of A .

Set

$$\varepsilon_0 = \frac{\varepsilon}{12(n+1)^5}.$$

Choose $\delta > 0$ with $\delta < \varepsilon_0$, and so small that whenever e and f are projections in a C^* -algebra C such that $\|e - f\| < \delta$, then there is a partial isometry $s \in C$ such that

$$ss^* = e, \quad s^*s = f, \quad \text{and} \quad \|s - e\| < \varepsilon_0.$$

Since A is AF, there is a finite dimensional C^* -subalgebra $E_0 \subset A$ such that for every a in the finite set

$$S_0 = F \cup \alpha(F) \cup \dots \cup \alpha^{n-1}(F),$$

there exists $b \in E_0$ such that $\|a - b\| < \delta$. Apply Lemma 11.1 with S_0 in place of F , with E_0 in place of the finite dimensional subalgebra E , and with δ in place of ε . We obtain a unitary $y \in A$ and mutually orthogonal projections $e_0, e_1, \dots, e_{n-1} \in A$ which commute with all elements of yE_0y^* , such that $\|e_j a - a e_j\| < \delta$ for all $a \in S_0$, such that $\sum_{j=0}^{n-1} e_j = 1$, such that $\|\alpha(e_j) - e_{j+1}\| < \delta$, and such that $\|y - 1\| < \delta$.

According to the choice of δ , for $1 \leq j \leq n-1$ there are partial isometries $w_j \in A$ such that

$$w_j w_j^* = e_j, \quad w_j^* w_j = \alpha(e_{j-1}), \quad \text{and} \quad \|w_j - e_j\| < \varepsilon_0.$$

Apply Lemma 2.6 to the e_j and w_j , with ε_0 in place of ε . We obtain a unitary w as there such that $\|w - 1\| < 2n^2\varepsilon_0$ and satisfying the condition of Lemma 2.4, such that moreover the automorphism $\beta = \text{Ad}(w) \circ \alpha$ satisfies $\beta^n = \text{id}_A$,

$$\|\beta^k(a) - \alpha^k(a)\| \leq 4kn^2\varepsilon_0\|a\|$$

for $a \in A$, and $\beta(e_j) = e_{j+1}$ for all j .

Define

$$E = \bigoplus_{k=0}^{n-1} \beta^k(e_0yE_0y^*e_0) = \bigoplus_{k=0}^{n-1} e_k\beta^k(yE_0y^*)e_k.$$

Since the e_k are orthogonal, $\sum_{k=0}^{n-1} e_k = 1$, and e_0 commutes with every element of yE_0y^* , it follows that E is a β -invariant finite dimensional subalgebra of A such that $1_A \in E$.

Let $a \in F$. We estimate the distance from a to E . We begin by estimating, using $\alpha^{n-k}(a) \in S_0$ at the fourth step,

$$\begin{aligned} \| [e_k, a] \| &= \| \beta^{n-k}([e_k, a]) \| = \| [e_0, \beta^{n-k}(a)] \| \\ &\leq 2\| \beta^{n-k}(a) - \alpha^{n-k}(a) \| + \| [e_0, \alpha^{n-k}(a)] \| \\ &< 8(n-k)n^2\varepsilon_0 + \delta < (8n^3 + 1)\varepsilon_0. \end{aligned}$$

It follows that if $k \neq l$ then $\|e_kae_l\| = \| [e_k, a]e_l \| < (8n^3 + 1)\varepsilon_0$. Since there are fewer than n^2 terms in the sum in the second expression, we can estimate

$$\left\| a - \sum_{k=0}^{n-1} e_kae_k \right\| \leq \sum_{k=0}^{n-1} \sum_{l \neq k} \|e_kae_l\| < n^2(8n^3 + 1)\varepsilon_0.$$

Moreover, by construction there exists $b_k \in E_0$ such that $\|b_k - \alpha^{n-k}(a)\| < \delta$. Then

$$\|b_k - \beta^{n-k}(a)\| < \delta + 4(n-k)n^2\varepsilon_0 < (4n^3 + 1)\varepsilon_0,$$

whence

$$\| \beta^k(e_0b_ke_0) - e_kae_k \| = \| e_0b_ke_0 - e_0\beta^{n-k}(a)e_0 \| < (4n^3 + 1)\varepsilon_0.$$

It follows that $c = \sum_{k=0}^{n-1} \beta^k(e_0b_ke_0) \in E$ and satisfies

$$\begin{aligned} \|a - c\| &\leq \left\| a - \sum_{k=0}^{n-1} e_kae_k \right\| + \sum_{k=0}^{n-1} \|e_kae_k - \beta^k(e_0b_ke_0)\| \\ &< n^2(8n^3 + 1)\varepsilon_0 + n(4n^3 + 1)\varepsilon_0 \leq 12(n+1)^5\varepsilon_0 < \varepsilon. \end{aligned}$$

Let

$$D_0 = C^*(\mathbf{Z}_n, E, \beta|_E),$$

which is a finite dimensional subalgebra of $C^*(\mathbf{Z}_n, A, \beta)$. Let

$$\varphi: C^*(\mathbf{Z}_n, A, \beta) \rightarrow C^*(\mathbf{Z}_n, A, \alpha)$$

be the isomorphism of Lemma 2.4. We take the finite dimensional subalgebra D to be $D = \varphi(D_0)$. Because $\varphi(a) = a$ for $a \in A$, we have shown that for every $a \in F$ there is $c \in E \subset D_0$ such that $\|c - a\| < \varepsilon$. Let $v \in C^*(\mathbf{Z}_n, A, \beta)$ be the canonical unitary implementing the automorphism β . Then $\varphi(v) \in D$ and Lemma 2.4 gives $\|\varphi(v) - u\| = \|w - 1\| < 2n^2\varepsilon_0 < \varepsilon$. ■

The following two lemmas give essentially the only restrictions we know on the behavior of tracially approximately inner automorphisms: they act as the identity on K_0 mod infinitesimals and on the tracial states. It is also easy to see that if $\alpha \in \text{Aut}(A)$ is tracially approximately inner but not approximately inner, then there must be arbitrarily small positive elements $\eta \in K_0(A)$ such that $\alpha_*(\eta) = \eta$. The dual automorphisms of the actions in Examples 12.3 and 12.11 are tracially approximately inner, but the one in Example 12.3 is not the identity on K_0 and the one in Example 12.11 is not the identity on K_1 .

Recall that an element η of a partially ordered group G with order unit $u \in G_+ \setminus \{0\}$ is *infinitesimal* if $-mu \leq n\eta \leq mu$ for all $m, n \in \mathbf{N}$ with $m > 0$. See Definition 1.10 of [25], where this definition is given for simple dimension groups. Clearly we need only consider $m = 1$. By Proposition 4.7 of [26], an equivalent condition is that all states on (G, u) vanish on η .

Lemma 11.3. Let A be a stably finite simple unital C^* -algebra, and let $\alpha \in \text{Aut}(A)$ be tracially approximately inner. Then $\alpha_*(\eta) - \eta$ is infinitesimal for every $\eta \in K_0(A)$.

Proof. We prove that for every $\eta \in K_0(A)$ we have $-[1_A] \leq \alpha_*(\eta) - \eta \leq [1_A]$. This implies the result, because replacing η by $n\eta$ gives $-[1_A] \leq n[\alpha_*(\eta) - \eta] \leq [1_A]$.

Accordingly, let $\eta \in K_0(A)$, and choose $n \in \mathbf{N}$ and projections $p, r \in M_n(A)$ such that $\eta = [p] - [r]$. Let $\chi: \mathbf{R} \setminus \{\frac{1}{2}\} \rightarrow \mathbf{R}$ be the characteristic function of $(\frac{1}{2}, \infty)$. Choose $\varepsilon > 0$ so small that $(n^2 + 2)\varepsilon < \frac{1}{6}$, and also so small that whenever C is a unital C^* -algebra and $f, p \in C$ are projections such that $\|f q f - q f\| < (n^2 + 2)\varepsilon$, then $\frac{1}{2}$ is not in the spectrum of either $f q f$ or $(1 - f)q(1 - f)$, and moreover the projections $q_0 = \chi(f q f)$ and $q_1 = \chi((1 - f)q(1 - f))$ satisfy $\|q_0 + q_1 - q\| < \frac{1}{6}$.

Apply Definition 3.1 with $F = \{p_{j,k}, r_{j,k}: 1 \leq j, k \leq n\}$, the set of all matrix entries of p and r , with ε as just chosen, with $N = 2n$, and with $x = 1$. Let $e \in A$ and $v \in eAe$ be the resulting projection and unitary.

We have

$$\|(1 \otimes e)p - p(1 \otimes e)\| \leq \sum_{j, k=1}^n \|ep_{j,k} - p_{j,k}e\| < n^2\varepsilon.$$

By the choice of ε the projections

$$p_0 = \chi((1 \otimes e)p(1 \otimes e)) \in M_n(eAe)$$

and

$$p_1 = \chi((1 - 1 \otimes e)p(1 - 1 \otimes e)) \in M_n((1 - e)A(1 - e))$$

are defined and satisfy $\|p_0 + p_1 - p\| < \frac{1}{6}$. To get a similar result for $(\text{id} \otimes \alpha)(p)$, we begin by observing that

$$\|(\text{id} \otimes \alpha)^{-1}(1 \otimes e) - 1 \otimes e\| = \|e - \alpha(e)\| < \varepsilon.$$

So

$$\begin{aligned} &\|(1 \otimes e)[(\text{id} \otimes \alpha)(p)] - [(\text{id} \otimes \alpha)(p)](1 \otimes e)\| \\ &= \|[(\text{id} \otimes \alpha)^{-1}(1 \otimes e)]p - p[(\text{id} \otimes \alpha)^{-1}(1 \otimes e)]\| \\ &\leq 2\|(\text{id} \otimes \alpha)^{-1}(1 \otimes e) - 1 \otimes e\| + \|(1 \otimes e)p - p(1 \otimes e)\| \\ &< 2\varepsilon + n^2\varepsilon = (n^2 + 2)\varepsilon. \end{aligned}$$

Therefore we get projections $q_0 \in M_n(eAe)$ and $q_1 \in M_n((1-e)A(1-e))$ such that $\|q_0 + q_1 - (\text{id} \otimes \alpha)(p)\| < \frac{1}{6}$. Note in particular that

$$[p] = [p_0] + [p_1] \quad \text{and} \quad \alpha_*([p]) = [q_0] + [q_1]$$

in $K_0(A)$.

We now claim that $[p_0] = [q_0]$ in $K_0(A)$. First,

$$\|(1 \otimes e)p(1 \otimes e) - p_0\| = \|(1 \otimes e)[p - (p_0 + p_1)](1 \otimes e)\| < \frac{1}{6}.$$

Next,

$$\begin{aligned} & \|(\text{id} \otimes \alpha)((1 \otimes e)p(1 \otimes e)) - q_0\| \\ & \leq 2\|(\text{id} \otimes \alpha)(1 \otimes e) - 1 \otimes e\| + \|(1 \otimes e)(\text{id} \otimes \alpha)(p)(1 \otimes e) - q_0\| \\ & < 2\varepsilon + \|(\text{id} \otimes \alpha)(p) - (q_0 + q_1)\| < 2\varepsilon + \frac{1}{6}. \end{aligned}$$

Finally,

$$\begin{aligned} & \|(1 \otimes v)(1 \otimes e)p(1 \otimes e)(1 \otimes v)^* - (\text{id} \otimes \alpha)((1 \otimes e)p(1 \otimes e))\| \\ & \leq \sum_{j,k=1}^n \|vep_{j,k}ev^* - \alpha(ep_{j,k}e)\| < n^2\varepsilon. \end{aligned}$$

Putting these estimates together gives

$$\|(1 \otimes v)p_0(1 \otimes v)^* - q_0\| < \frac{1}{6} + n^2\varepsilon + 2\varepsilon + \frac{1}{6} < \frac{1}{2}.$$

The claim follows.

Repeating the argument of the last two paragraphs with r in place of p , we find projections

$$r_0, s_0 \in M_n(eAe) \quad \text{and} \quad r_1, s_1 \in M_n((1-e)A(1-e))$$

such that

$$[r] = [r_0] + [r_1], \quad \alpha_*([r]) = [s_0] + [s_1], \quad \text{and} \quad [r_0] = [s_0]$$

in $K_0(A)$.

Now recall from Condition (5) of Definition 3.1 that there are $2n$ mutually orthogonal projections $f_1, f_2, \dots, f_{2n} \leq e$, each of which is Murray-von Neumann equivalent to $1 - e$. Since $p_1, q_1, r_1, s_1 \in M_n((1-e)A(1-e))$, this implies that

$$[p_1] + [s_1] \leq [1_A] \quad \text{and} \quad [q_1] + [r_1] \leq [1_A]$$

in $K_0(A)$. Therefore

$$\alpha_*(\eta) - \eta = \alpha_*([p]) - [p] - \alpha_*([r]) + [r] = [q_1] - [p_1] - [s_1] + [r_1],$$

and

$$-[1_A] \leq -[p_1] - [s_1] \leq \alpha_*(\eta) - \eta \leq [q_1] + [r_1] \leq [1_A].$$

This completes the proof. ■

Lemma 11.4. Let A be a simple unital C*-algebra, and let $\alpha \in \text{Aut}(A)$ be tracially approximately inner. Then $\tau \circ \alpha = \tau$ for every tracial state τ on A .

Proof. Let $a \in A$ and let $\varepsilon > 0$. We show that $|\tau(\alpha(a)) - \tau(a)| < \varepsilon$. Without loss of generality $a \geq 0$ and $\|a\| \leq 1$. Choose $N \in \mathbf{N}$ such that $\frac{1}{N} < \frac{1}{3}\varepsilon$. Apply Definition 3.1 with $F = \{a\}$, with $\frac{1}{3}\varepsilon$ in place of ε , with N as chosen here, and with $x = 1$. Let $e \in A$ and $v \in eAe$ be the resulting projection and unitary. Condition (5) of Definition 3.1 implies that $\tau(1 - e) \leq \frac{1}{N} < \frac{1}{3}\varepsilon$. We have

$$\tau(a) = \tau(eae) + \tau((1 - e)a(1 - e)),$$

with

$$0 \leq \tau((1 - e)a(1 - e)) \leq \tau(1 - e) < \frac{1}{3}\varepsilon,$$

so $|\tau(a) - \tau(eae)| < \frac{1}{3}\varepsilon$. Similarly $\tau(\alpha(1 - e)) \leq \frac{1}{N} < \frac{1}{3}\varepsilon$ and $|\tau(\alpha(a)) - \tau(\alpha(eae))| < \frac{1}{3}\varepsilon$. Finally, $\|veaev^* - \alpha(eae)\| < \frac{1}{3}\varepsilon$ and $\tau(veaev^*) = \tau(eae)$. Putting these together gives

$$\begin{aligned} |\tau(\alpha(a)) - \tau(a)| &\leq |\tau(\alpha(a)) - \tau(\alpha(eae))| + |\tau(\alpha(eae)) - \tau(eae)| + |\tau(eae) - \tau(a)| \\ &< \frac{1}{3}\varepsilon + \frac{1}{3}\varepsilon + \frac{1}{3}\varepsilon = \varepsilon, \end{aligned}$$

as desired. ■

12. EXAMPLES

We look at several examples of actions on the 2^∞ UHF algebra. They demonstrate the following:

- Even on a UHF algebra, an action with the tracial Rokhlin property need not have the strict Rokhlin property, and in fact the crossed product by such an action need not be AF. See Example 12.11.
- If an automorphism α of a simple unital C*-algebra A with tracial rank zero is approximately inner and generates an action of \mathbf{Z}_n with the tracial Rokhlin property, it does not follow that the automorphisms of the dual action are approximately inner—even if A is UHF and α is the pointwise limit of inner automorphisms $\text{Ad}(u_k)$ with $u_k^n = 1$. See Example 12.3.
- If an automorphism α of a simple unital C*-algebra A with tracial rank zero is tracially approximately inner and generates an action of \mathbf{Z}_n with the strict Rokhlin property, it does not follow that the automorphisms of the dual action are approximately inner—even if A is AF. Use the dual of the action in Example 12.11.
- A tracially approximately inner automorphism of a simple unital C*-algebra A with tracial rank zero need not be trivial on $K_0(A)$ —even if A is AF and α generates an action of \mathbf{Z}_n with the strict Rokhlin property. Use the dual of the action in Example 12.3.
- A tracially approximately inner automorphism of a simple unital C*-algebra A with tracial rank zero need not be trivial on $K_1(A)$ —even if A is AT and α generates an action of \mathbf{Z}_n with the strict Rokhlin property. Use the dual of the action in Example 12.11.
- There is an automorphism α of a simple AF algebra such that $\alpha^n = \text{id}_A$, such that $C^*(\mathbf{Z}_n, A, \alpha)$ is again a simple AF algebra, but such that this action does not have the tracial Rokhlin property. This can happen even when the dual action has the strict Rokhlin property and α is approximately inner. See Example 12.7.

- The dual of an action with the tracial Rokhlin property need not have the tracial Rokhlin property, even when both the original algebra and the crossed product are simple AF algebras, and even when the original action also has the strict Rokhlin property. Use the dual of the action in Example 12.7.

The first three examples are product type actions of \mathbf{Z}_2 on the 2^∞ UHF algebra D . In each case we represent D as an infinite tensor product, and the automorphism generating the action as an infinite tensor product of inner automorphisms. It is useful to give a general lemma.

Lemma 12.1. Let D be an infinite tensor product C*-algebra and let $\alpha \in \text{Aut}(D)$ be an automorphism of order two, of the form

$$D = \bigotimes_{n=1}^{\infty} M_{k(n)} \quad \text{and} \quad \alpha = \bigotimes_{n=1}^{\infty} \text{Ad}(e_n - f_n),$$

with $k(n) \in \mathbf{N}$ and where $e_n, f_n \in M_{k(n)}$ are projections with $e_n + f_n = 1$. Let $D_n = \bigotimes_{m=1}^n M_{k(m)}$, so that $D = \varinjlim D_n$, and write $D_n = M_{t(n)}$, where $t(n) = \prod_{m=1}^n k(m)$. Then the direct system of crossed products can be identified as

$$C^*(\mathbf{Z}_2, D_n) \cong M_{t(n)} \oplus M_{t(n)},$$

with the dual action given by the flip $\sigma_n(a, b) = (b, a)$ for $a, b \in M_{t(n)}$, and where the maps

$$\psi_n: M_{t(n-1)} \oplus M_{t(n-1)} \rightarrow M_{t(n)} \oplus M_{t(n)}$$

are given by

$$\psi_n(a, b) = (a \otimes e_n + b \otimes f_n, b \otimes e_n + a \otimes f_n)$$

for $a, b \in M_{t(n-1)}$. Assuming moreover that $e_n, f_n \neq 0$ for all n , we have:

- (1) $C^*(\mathbf{Z}_2, D, \alpha)$ is a simple unital AF algebra.
- (2) The action of \mathbf{Z}_2 generated by α is approximately representable in the sense of Definition 3.6(2) of [29].
- (3) The dual action on $C^*(\mathbf{Z}_2, D, \alpha)$ has the strict Rokhlin property.
- (4) The action of \mathbf{Z}_2 generated by α has the strict Rokhlin property if and only if the dual action is approximately representable.
- (5) If the action of \mathbf{Z}_2 generated by α has the tracial Rokhlin property, then the generating automorphism $\hat{\alpha}$ of the dual action is tracially approximately inner.

Proof. For $n \in \mathbf{N}$ and a unitary $v \in M_n$ with $v^2 = 1$, we use the isomorphism $C^*(\mathbf{Z}_2, M_n, \text{Ad}(v)) \rightarrow M_n \oplus M_n$ which sends $a \in M_n$ to (a, a) and the canonical unitary of the crossed product to $(v, -v)$. The identification of the direct system of crossed products is then a calculation, which we omit. Now assume that $e_n, f_n \neq 0$ for all n . Simplicity of $C^*(\mathbf{Z}_2, D, \alpha)$ follows from the fact that the partial embedding multiplicities in the direct system of crossed products are all nonzero, and the rest of (1) is immediate. Part (2) is immediate. Part (3) now follows from Lemma 3.8(2) of [29]. Part (4) is Lemma 3.8(1) of [29]. To prove Part (5), and we use Theorem 4.3. The algebra D has cancellation of projections because it is AF, and $C^*(\mathbf{Z}_2, D, \alpha)$ has the weak divisibility property (Definition 4.1) because it is also AF. Therefore Theorem 4.3 applies. ■

Presumably the converse of Part (5) is true as well, but we have no need for it.

Our first example is the most regular possible, and is given to contrast with the remaining ones.

Example 12.2. Let α be the automorphism of order 2 given by

$$D = \bigotimes_{n=1}^{\infty} M_2 \quad \text{and} \quad \alpha = \bigotimes_{n=1}^{\infty} \text{Ad} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Then the action of \mathbf{Z}_2 generated by α has the strict Rokhlin property, the crossed product is again the 2^{∞} UHF algebra, and the dual action is just another copy of the given action. All this is easily proved using Lemma 12.1, and is also a special case of Example 3.2 of [29].

Example 12.3. Let β be the automorphism of order 2 given by

$$D = \bigotimes_{n=1}^{\infty} M_{2^n} \quad \text{and} \quad \alpha = \bigotimes_{n=1}^{\infty} \text{Ad}(1_{2^{n-1}+1} \oplus (-1_{2^{n-1}-1})).$$

The automorphism in the n -th tensor factor is conjugation by a diagonal unitary in which $2^{n-1} + 1$ diagonal entries are equal to 1 and $2^{n-1} - 1$ diagonal entries are equal to -1 . The crossed product $C^*(\mathbf{Z}_2, D, \beta)$ is a simple unital AF algebra by Lemma 12.1(1).

In this case, the action of \mathbf{Z}_2 generated by β has the tracial Rokhlin property, but does not have the strict Rokhlin property. The dual action has the strict Rokhlin property and its generator is tracially approximately inner, but the generator is not approximately inner and does not induce the identity map on $K_0(C^*(\mathbf{Z}_2, D, \beta))$. We prove this in the next three propositions.

Proposition 12.4. The action of \mathbf{Z}_2 generated by the automorphism β of Example 12.3 has the tracial Rokhlin property.

Proof. We verify the conditions of Theorem 8.2. (We will not use the full strength of this theorem because the elements we construct will in fact be projections.) Let $D_n = \bigotimes_{k=1}^n M_{2^k}$, so that $D_n = D_{n-1} \otimes M_{2^n}$ and $D = \varinjlim D_n$. Let

$$v_n = 1_{2^{n-1}+1} \oplus (-1_{2^{n-1}-1}) \in M_{2^n} \quad \text{and} \quad u_n = \bigotimes_{k=1}^n v_n,$$

so that for $a \in D_n$ we have $\beta(a) = u_n a u_n^*$.

Let $F \subset D$ be finite and let $\varepsilon > 0$. Choose m and a finite set $S \subset D_m$ such that every element of F is within $\frac{1}{2}\varepsilon$ of an element of S . Choose $n > m$ and so large that $2^{-m+1} < \varepsilon$. Define $p = \text{diag}(0, 1, 1, \dots, 1) \in M_{2^{m-1}}$. Take

$$q_0 = \frac{1}{2} \begin{pmatrix} p & p \\ p & p \end{pmatrix} \quad \text{and} \quad q_1 = \frac{1}{2} \begin{pmatrix} p & -p \\ -p & p \end{pmatrix},$$

and for $j = 0, 1$ define

$$e_j = 1_{D_{m-1}} \otimes q_j \in D_m \subset D.$$

Then each e_j is a projection, and $e_0 e_1 = 0$. Moreover, e_j commutes exactly with every element of S (indeed, with every element of D_n), so that $\|e_j a - a e_j\| < \varepsilon$ for all $a \in F$.

To compute $\beta(e_j)$, we write

$$v_n = \begin{pmatrix} 1 & 0 \\ 0 & 1 - 2p \end{pmatrix}.$$

Then calculations show that $v_n q_0 v_n^* = q_1$. Consequently $\beta(e_0) = e_1$. Finally, the unique tracial state τ on D restricts to the usual tracial state on $1_{D_{m-1}} \otimes M_{2^m} \otimes 1 \subset D$, so

$$\tau(1 - e_0 - e_1) = \frac{2 \cdot \text{rank}(p)}{2^m} = \frac{1}{2^{m-1}}$$

satisfies $|\tau(1 - e_0 - e_1)| < \varepsilon$. This completes the verification of the hypotheses of Theorem 8.2, so the action generated by β has the tracial Rokhlin property. ■

Proposition 12.5. The action of \mathbf{Z}_2 generated by the automorphism β of Example 12.3 does not have the strict Rokhlin property.

Proof. We show that there is no projection $e \in D$ such that $\|\beta(e) - (1 - e)\| < 1$. Let D_n , v_n , and u_n be as in the proof of Proposition 12.4. Write $D_n = M_{t(n)}$, where $t(n) = 2^{n(n+1)/2}$. Then u_n is a $t(n) \times t(n)$ diagonal matrix whose diagonal entries are all either 1 or -1 . Let $r(n)$ be the number of entries equal to 1, and let $s(n)$ be the number of entries equal to -1 . We prove by induction that $r(n) - s(n) = 2^n$. This is certainly true for $n = 1$, when $t(n) = 2$ and $u_n = 1$. If it is true for n , then

$$r(n+1) = (2^n + 1)r(n) + (2^n - 1)s(n)$$

and

$$s(n+1) = (2^n - 1)r(n) + (2^n + 1)s(n),$$

so

$$r(n+1) - s(n+1) = 2(r(n) - s(n)) = 2^{n+1}.$$

This completes the induction.

Now suppose there is a projection $e \in D$ such that $\|\beta(e) - (1 - e)\| < 1$. Set $\varepsilon = \frac{1}{2}(1 - \|\beta(e) - (1 - e)\|) > 0$. Choose n and a projection $f \in D_n$ such that $\|e - f\| < \varepsilon$. Then $\|u_n f u_n^* - (1 - f)\| < 1$. It follows that $f \sim 1 - f$. Now recall that $D_n = M_{t(n)}$, and represent this algebra on the Hilbert space $\mathbf{C}^{t(n)}$ in the usual way. Then $\text{rank}(f) = \frac{1}{2}t(n)$. We can write $u_n = q_0 - q_1$ where q_0 and q_1 are orthogonal projections of ranks $r(n)$ and $s(n)$. From the previous paragraph, $r(n) > \frac{1}{2}t(n)$, so $E = q_0 \mathbf{C}^{t(n)} \cap f \mathbf{C}^{t(n)}$ is nontrivial. Choose $\xi \in E$ with $\|\xi\| = 1$. Then

$$(1 - f)\xi = 0, \quad q_0\xi = \xi, \quad \text{and} \quad q_1\xi = 0,$$

so

$$[u_n f u_n^* - (1 - f)]\xi = \xi.$$

It follows that $\|u_n f u_n^* - (1 - f)\| \geq 1$. This contradiction shows that e does not exist, and that the action generated by β does not have the strict Rokhlin property. ■

Proposition 12.6. Let $\beta \in \text{Aut}(D)$ be as in Example 12.3. Let $\widehat{\beta}$ be the nontrivial automorphism of the dual action on $C^*(\mathbf{Z}_2, D, \beta)$. Then $\widehat{\beta}$ is tracially approximately inner and generates an action with the strict Rokhlin property, but $\widehat{\beta}$ is not approximately inner and is nontrivial on $K_0(C^*(\mathbf{Z}_2, D, \beta))$.

Proof. It follows from Lemma 12.1(5) and Proposition 12.4 that $\widehat{\beta}$ is tracially approximately inner, and from Lemma 12.1(3) that $\widehat{\beta}$ generates an action with the strict Rokhlin property.

We show that $\widehat{\beta}$ is nontrivial on $K_0(C^*(\mathbf{Z}_2, D, \beta))$; that $\widehat{\beta}$ is not approximately inner follows. Using Lemma 12.1, we can identify $K_0(C^*(\mathbf{Z}_2, D, \beta))$ as the direct limit of the system

$$\mathbf{Z}^2 \xrightarrow{T_1} \mathbf{Z}^2 \xrightarrow{T_2} \mathbf{Z}^2 \xrightarrow{T_3} \dots,$$

with

$$T_n = \begin{pmatrix} 2^{n-1} + 1 & 2^{n-1} - 1 \\ 2^{n-1} - 1 & 2^{n-1} + 1 \end{pmatrix},$$

and where $\widehat{\beta}_*$ is the direct limit of the maps $(j, k) \mapsto (k, j)$ on \mathbf{Z}^2 . One proves by induction that

$$T_n \circ T_{n-1} \circ \dots \circ T_1(1, -1) = (2^n, -2^n).$$

This expression is nonzero for all n , so the image of $(1, -1)$ is a nonzero element η of $K_0(C^*(\mathbf{Z}_2, D, \beta))$. The element η is not torsion, since $C^*(\mathbf{Z}_2, D, \beta)$ is an AF algebra. Evidently $\widehat{\beta}_*(\eta) = -\eta \neq \eta$. ■

The following example was suggested by Izumi. Some of the properties given here are folklore, but we have been unable to find a reference for the proofs. This example is a special case of an example used for other purposes in Example 3.14 of [28].

Example 12.7. Let γ be the automorphism of order 2 given by

$$D = \bigotimes_{n=1}^{\infty} M_{2^n} \quad \text{and} \quad \gamma = \bigotimes_{n=1}^{\infty} \text{Ad}(1_{2^n-1} \oplus (-1)).$$

The automorphism in the n -th tensor factor is conjugation by a diagonal unitary in which $2^n - 1$ diagonal entries are equal to 1 and one diagonal entry is equal to -1 .

We prove the following facts in the next three propositions. The automorphism γ is approximately inner, the action of \mathbf{Z}_2 it generates does not have the tracial Rokhlin property, and the generator of the dual action is not tracially approximately inner. Nevertheless, the crossed product is a simple unital AF algebra, and the dual action has the strict Rokhlin property. As an “explanation”, in the factor representation of D associated to the trace, γ becomes inner.

Proposition 12.8. Let $\gamma \in \text{Aut}(D)$ be as in Example 12.7. Then the crossed product $C^*(\mathbf{Z}_2, D, \gamma)$ is a simple AF algebra which has exactly two extreme tracial states. The dual action exchanges these two tracial states.

Proof. Let $D_n = \bigotimes_{k=1}^n M_{2^k}$, so that $D_n = D_{n-1} \otimes M_{2^n}$ and $D = \varinjlim D_n$. We identify the direct system of crossed products as in Lemma 12.1, and we follow the notation there, with

$$e_n = 1_{2^n-1} \oplus 0 \in M_{2^n} \quad \text{and} \quad f_n = 0_{2^n-1} \oplus 1 \in M_{2^n}.$$

and with $t(n) = 2^{n(n+1)/2}$. The algebra $C^*(\mathbf{Z}_2, D, \gamma)$ is a simple AF algebra by Lemma 12.1(1).

We want to identify all tracial states on $C^*(\mathbf{Z}_2, D, \gamma)$, but we begin with some convenient notation. For $\lambda \in \mathbf{R}$ define the matrix

$$T(\lambda) = \frac{1}{2} \begin{pmatrix} 1+\lambda & 1-\lambda \\ 1-\lambda & 1+\lambda \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

From the second expression, we see that $T(\lambda\mu) = T(\lambda)T(\mu)$ for $\lambda, \mu \in \mathbf{R}$, and that $T(1) = 1$. From the first expression, we see that the matrix of partial embedding multiplicities of ψ_n is exactly $2^n T(1 - \frac{1}{2^{n-1}})$. Moreover, for $\lambda \in [0, 1]$, if $(r_0, s_0), (r, s) \in \mathbf{R}^2$ satisfy $T(\lambda)(r_0, s_0) = (r, s)$, and if $r_0, s_0 \in [0, 1]$ with $r_0 + s_0 = 1$, then $r, s \in [0, 1]$ with $r + s = 1$.

Let tr_m denote the normalized trace on M_m . Tracial states on $C^*(\mathbf{Z}_2, D, \gamma)$ are in one to one correspondence with sequences $(\tau_n)_{n \in \mathbf{N}}$ of tracial states τ_n on $M_{t(n)} \oplus M_{t(n)}$ satisfying the compatibility conditions $\tau_n \circ \psi_n = \tau_{n-1}$ for all n . The tracial state τ_n has the form $\tau_n(a, b) = r_n \text{tr}_{t(n)}(a) + s_n \text{tr}_{t(n)}(b)$ for $r_n, s_n \in [0, 1]$ with $r_n + s_n = 1$, and the compatibility condition is exactly $T(1 - \frac{1}{2^{n-1}})(r_n, s_n) = (r_{n-1}, s_{n-1})$ in \mathbf{R}^2 .

Define

$$\lambda_n = \prod_{k=n+1}^{\infty} \left(1 - \frac{1}{2^{k-1}}\right) \in [0, 1).$$

We claim that a sequence $(r_n, s_n)_{n \in \mathbf{N}}$ corresponds to a tracial state on the crossed product $C^*(\mathbf{Z}_2, D, \gamma)$ if and only if there is $r \in [0, 1]$ such that for all $n \in \mathbf{N}$ we have $(r_n, s_n) = T(\lambda_n)(r, 1-r)$. One direction is easy: given r , the sequence $(r_n, s_n)_{n \in \mathbf{N}}$ defined by $T(\lambda_n)(r, 1-r) = (r_n, s_n)$ clearly satisfies $r_n, s_n \in [0, 1]$ and $r_n + s_n = 1$, and the compatibility condition follows from the relation $(1 - \frac{1}{2^{n-1}})\lambda_n = \lambda_{n-1}$. For the converse, one observes that $\log(1-x) \geq -2x$ for $0 \leq x \leq \frac{1}{2}$, so that

$$\log(\lambda_n) = \sum_{k=n+1}^{\infty} \log\left(1 - \frac{1}{2^{k-1}}\right) \geq \sum_{k=n+1}^{\infty} 2\left(1 - \frac{1}{2^{k-1}}\right) > -\infty.$$

Therefore $\lambda_n > 0$, so that $T(\lambda_n)$ is invertible. Define $r, s \in \mathbf{R}$ by $(r, s) = T(\lambda_n)^{-1}(r_n, s_n)$. The compatibility condition guarantees that this definition does not depend on n . Moreover, $\lim_{n \rightarrow \infty} \lambda_n = 1$, whence $\lim_{n \rightarrow \infty} T(\lambda_n)^{-1} = 1$. Therefore $\lim_{n \rightarrow \infty} r_n = r$ and $\lim_{n \rightarrow \infty} s_n = s$. It follows that $r, s \in [0, 1]$ and $r + s = 1$. This completes the proof of the claim.

Since $T(\lambda_n)$ is invertible, we have an affine parametrization of the tracial states on $C^*(\mathbf{Z}_2, D, \gamma)$ by $[0, 1]$, from which it is clear that there are exactly two extreme tracial states. That the dual action exchanges them is clear from the identification of the dual action with the flip in Lemma 12.1. ■

Proposition 12.9. Let $\gamma \in \text{Aut}(D)$ be as in Example 12.7. Then:

- (1) The action of \mathbf{Z}_2 generated by γ does not have the tracial Rokhlin property.
- (2) The automorphism γ is approximately inner.
- (3) The dual action on $C^*(\mathbf{Z}_2, D, \gamma)$ has the strict Rokhlin property.
- (4) The generating automorphism $\widehat{\gamma}$ of the dual action is not tracially approximately inner.

Proof. Proposition 12.8 shows that $\widehat{\gamma}$ is not the identity on the tracial state space. So Lemma 11.4 implies that $\widehat{\gamma}$ is not tracially approximately inner. This is (4). The rest follows from Lemma 12.1. ■

Proposition 12.10. Let $\gamma \in \text{Aut}(D)$ be as in Example 12.7. Let π be the Gelfand-Naimark-Segal representation associated with the unique tracial state τ on D . Then the automorphism $\bar{\gamma}$ of $\pi(D)''$ induced by γ is inner.

Proof. In a slight modification of the notation used previously, set

$$e_n^{(0)} = 1_{2^n-1} \oplus 0 \in M_{2^n} \quad \text{and} \quad f_n^{(0)} = 0_{2^n-1} \oplus 1 \in M_{2^n}.$$

Then define projections $e_n, f_n \in D = \bigotimes_{n=1}^{\infty} M_{2^n}$ by

$$e_n = 1 \otimes \cdots 1 \otimes e_n^{(0)} \otimes 1 \otimes \cdots \quad \text{and} \quad f_n = 1 \otimes \cdots 1 \otimes f_n^{(0)} \otimes 1 \otimes \cdots,$$

where the nontrivial tensor factors are in the n -th positions. Define unitaries in D by $v_n = e_n - f_n$ and $u_n = \prod_{k=1}^n v_k$. Further note that $\tau(f_n) = \frac{1}{2^n}$.

We show that $\lim_{n \rightarrow \infty} \pi(u_n)$ exists in $\pi(D)''$ in the strong operator topology, and that the limit is a unitary which implements γ on $\pi(D)$. This is easily seen to imply the result. Let ξ be the standard cyclic vector for the representation π , and let H be the Hilbert space on which it acts. For $a \in D$ we have

$$\begin{aligned} \|\pi(u_n)\pi(a)\xi - \pi(u_{n-1})\pi(a)\xi\|^2 &= \|\pi(v_n)\pi(a)\xi - \pi(a)\xi\|^2 \\ &= \langle \pi(a^*(v_n - 1)^*(v_n - 1)a)\xi, \xi \rangle \\ &= \tau(a^*(v_n - 1)^*(v_n - 1)a) \\ &= \tau((v_n - 1)a a^*(v_n - 1)^*) \\ &\leq \tau((v_n - 1)(v_n - 1)^*)\|a\|^2 = 4\tau(f_n)\|a\|^2. \end{aligned}$$

Since $\sum_{n=1}^{\infty} \tau(f_n)^{1/2} < \infty$, it follows that $(\pi(u_n)\pi(a)\xi)_{n \in \mathbb{N}}$ is a Cauchy sequence in H , and therefore converges. The elements $\pi(a)\xi$ form a dense subset of H , and $\sup_{n \in \mathbb{N}} \|\pi(u_n)\| < \infty$, so a standard argument shows that $u\eta = \lim_{n \rightarrow \infty} \pi(u_n)\eta$ exists for all $\eta \in H$. Further, u is clearly isometric, hence bounded. Since u_n is selfadjoint and the selfadjoint elements are strong operator closed in $L(H)$, we get $u^* = u$. Since multiplication is jointly strong operator continuous on bounded sets, $u^2 = 1$. Therefore u is a selfadjoint unitary.

For any $a \in D$, we have $\lim_{n \rightarrow \infty} u_n a u_n = \lim_{n \rightarrow \infty} u_n a u_n^* = \gamma(a)$ in norm, and, again using joint strong operator continuity of multiplication on bounded sets, $\lim_{n \rightarrow \infty} u_n a u_n = u a u = u a u^*$ in the strong operator topology. Therefore $u a u^* = \gamma(a)$, as desired. ■

Example 12.11. Let A be the 2^∞ UHF algebra, and let α be the automorphism constructed in Section 5 of [3]. It follows from Corollary 5.3.2 of [3] and Takai duality that $C^*(\mathbf{Z}_2, A, \alpha)$ is not an AF algebra. We prove in Proposition 12.14 below that α generates an action of \mathbf{Z}_2 with the tracial Rokhlin property, and in Proposition 12.15 below that this action does not have the strict Rokhlin property, and that the generator of the dual action is tracially approximately inner but induces a nontrivial automorphism of K_1 . By construction, the action of \mathbf{Z}_2 generated by α is approximately representable in the sense of Definition 3.6(2) of [29]. (The construction is recalled below.) It follows from Lemma 3.8(2) of [29] that the dual action on $C^*(\mathbf{Z}_2, A, \alpha)$ has the strict Rokhlin property.

We remark that the methods used to prove the tracial Rokhlin property in this example seem likely to be more typical of proofs that actions on AH algebras have the tracial Rokhlin property than the methods used for Proposition 6.14 and Proposition 8.6.

We begin with a convenient description of the construction in [3], following Section 5 there. We make the convention that in any block matrix decomposition, all blocks are to be the same size, and we write 1_n for the identity of M_n . We take the identification of $M_m \otimes M_n$ with M_{mn} to send $a \otimes e_{j,k}$ to the $n \times n$ block matrix with $m \times m$ blocks, of which the (j, k) block is a and the rest are zero. Thus, $a \otimes 1 = \text{diag}(a, a, \dots, a)$. Also, we identify the circle S^1 with \mathbf{R}/\mathbf{Z} , and write elements of $C(S^1)$ as functions on $[0, 1]$ whose values at 0 and 1 are equal.

Following Definition 3.1.1 of [3], we choose a standard twice around embedding $\varphi^+ : C(S^1) \rightarrow C(S^1, M_2)$, given by choosing a continuous unitary path $c \in C([0, 1], M_2)$ with

$$c(0) = 1 \quad \text{and} \quad c(1) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

and then setting

$$\varphi^+(f)(t) = c(t) \begin{pmatrix} f(\frac{1}{2}t) & 0 \\ 0 & f(\frac{1}{2}(t+1)) \end{pmatrix} c(t)^*$$

for $f \in C(S^1)$. Further let $\varphi^- : C(S^1) \rightarrow C(S^1, M_2)$ be the standard -2 times around embedding $\varphi^-(f)(t) = \varphi^+(f)(1-t)$. Extend everything, using the same notation, to embeddings of $C(S^1, M_m)$ in $C(S^1, M_{2m})$, by using $\varphi^+ \otimes \text{id}_{M_m}$, etc.

Following Section 5 of [3], set $A_n = C(S^1, M_{4^n})$ and, remembering our convention on block sizes, define a unitary in $M_{4^n} \subset A_n$ by $u_n = \text{diag}(1, -1)$. Further define $\psi_n : A_n \rightarrow A_{n+1}$ by

$$\psi_n \begin{pmatrix} f_{1,1} & f_{1,2} \\ f_{2,1} & f_{2,2} \end{pmatrix} = \begin{pmatrix} \varphi^+(f_{1,1}) & 0 & \varphi^+(f_{1,2}) & 0 \\ 0 & \varphi^-(f_{2,2}) & 0 & \varphi^-(f_{2,1}) \\ \varphi^+(f_{2,1}) & 0 & \varphi^+(f_{2,2}) & 0 \\ 0 & \varphi^-(f_{1,2}) & 0 & \varphi^-(f_{1,1}) \end{pmatrix}.$$

Theorem 4.1.1, Proposition 5.1.1, and Proposition 5.1.2 of [3] show that the direct limit of the A_n using the maps ψ_n is the 2^∞ UHF algebra A , and that the automorphisms $\alpha_n = \text{Ad}(u_n)$ of A_n define an automorphism α of A of order two. It follows from Takai duality and Proposition 5.2.2 and Proposition 5.4.1 of [3] that $C^*(\mathbf{Z}_2, A, \alpha)$ is isomorphic to the tensor product of A and the 2^∞ Bunce-Deddens algebra.

Further let $\iota_n : M_{4^n} \rightarrow A_n$ be the embedding of matrices as constant functions, and define $\sigma_n : M_{4^n} \rightarrow M_{4^{n+1}}$ by

$$\iota_n \begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix} = \begin{pmatrix} a_{1,1} \otimes 1_2 & 0 & a_{1,2} \otimes 1_2 & 0 \\ 0 & a_{2,2} \otimes 1_2 & 0 & a_{2,1} \otimes 1_2 \\ a_{2,1} \otimes 1_2 & 0 & a_{2,2} \otimes 1_2 & 0 \\ 0 & a_{1,2} \otimes 1_2 & 0 & a_{1,1} \otimes 1_2 \end{pmatrix}.$$

It is a consequence of the next lemma that $\iota_{n+1} \circ \sigma_n = \psi_n \circ \iota_n$. Moreover, we get an automorphism μ_n of M_{4^n} by defining $\mu_n = \text{Ad}(u_n)$, and $\iota_n \circ \mu_n = \alpha_n \circ \iota_n$.

Lemma 12.12. Let the notation be as above. Let $\varepsilon \geq 0$, let $f \in A_n$, and let $a \in M_{4^n}$. Suppose $f(t) = a$ for all $t \in [\varepsilon, 1 - \varepsilon]$. Then $\psi_n(f)(t) = \sigma_n(a)$ for all $t \in [2\varepsilon, 1 - 2\varepsilon]$. In particular, $\iota_{n+1} \circ \sigma_n = \psi_n \circ \iota_n$.

Proof. Write

$$f = \begin{pmatrix} f_{1,1} & f_{1,2} \\ f_{2,1} & f_{2,2} \end{pmatrix} \quad \text{and} \quad a = \begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix}.$$

Then for each j and k , we have $f_{j,k}(t) = a_{j,k}$ for all $t \in [\varepsilon, 1-\varepsilon]$. For $t \in [2\varepsilon, 1-2\varepsilon]$ we therefore get

$$\begin{pmatrix} f_{j,k}(\frac{1}{2}t) & 0 \\ 0 & f_{j,k}(\frac{1}{2}(t+1)) \end{pmatrix} = \begin{pmatrix} a_{j,k} & 0 \\ 0 & a_{j,k} \end{pmatrix},$$

which commutes with $c(t)$. ■

In the next lemma we show, roughly, that whenever an element $f \in A_n = C(S^1, M_{4^n})$ is unitarily equivalent in $C([0, 1], M_{4^n})$, via invariant unitaries, to a function with small variation over intervals of length δ , then $\psi_n(f)$ is unitarily equivalent in $C([0, 1], M_{4^{n+1}})$, again via invariant unitaries, to a function with small variation over intervals of length 2δ .

Lemma 12.13. Let the notation be as above. Let $t \mapsto x(t)$ be a unitary element of $C([0, 1], M_{4^n})$ such that $\mu_n(x(t)) = x(t)$ for every $t \in [0, 1]$. Then there exists a unitary element $t \mapsto y(t)$ of $C([0, 1], M_{4^{n+1}})$ such that $\mu_{n+1}(y(t)) = y(t)$ for every $t \in [0, 1]$, with the property that whenever $\varepsilon > 0$, $\delta > 0$, and $f \in A_n$ satisfy

$$\|x(s)f(s)x(s)^* - x(t)f(t)x(t)^*\| < \varepsilon$$

for all $s, t \in [0, 1]$ such that $|s - t| < \delta$, then

$$\|y(s)\psi_n(f)(s)y(s)^* - y(t)\psi_n(f)(t)y(t)^*\| < \varepsilon$$

for all $s, t \in [0, 1]$ such that $|s - t| < 2\delta$.

Proof. The equation $\mu_n(x(t)) = x(t)$ implies that we can write $x(t) = x_1(t) \oplus x_2(t)$ for unitaries $x_1, x_2 \in C([0, 1], M_{2^{2n-1}})$. For $j = 1, 2$ define

$$y_j(t) = \begin{pmatrix} x_j(\frac{1}{2}t) & 0 \\ 0 & x_j(\frac{1}{2}(t+1)) \end{pmatrix} c(t)^*.$$

Then define

$$y(t) = \text{diag}(y_1(t), y_2(1-t), y_2(t), y_1(1-t)).$$

Evidently y is a unitary in $C([0, 1], M_{4^{n+1}})$ and $\mu_{n+1}(y(t)) = y(t)$ for every $t \in [0, 1]$.

To verify the conclusions of the lemma, it will simplify the notation to conjugate everything by the permutation matrix

$$w = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

(This conjugation is also used in Section 5 of [3].) Thus, let

$$\tilde{\psi}(f) = w\psi_n(f)w^* = \begin{pmatrix} \varphi^+(f_{1,1}) & \varphi^+(f_{1,2}) & 0 & 0 \\ \varphi^+(f_{2,1}) & \varphi^+(f_{2,2}) & 0 & 0 \\ 0 & 0 & \varphi^-(f_{1,1}) & \varphi^-(f_{1,2}) \\ 0 & 0 & \varphi^-(f_{2,1}) & \varphi^-(f_{2,2}) \end{pmatrix},$$

let

$$\tilde{u} = wu_{n+1}w^* = \text{diag}(1, -1, -1, 1),$$

let

$$\tilde{y}(t) = wy(t)w^* = \text{diag}(y_1(t), y_2(t), y_1(1-t), y_2(1-t)),$$

and similarly define $\tilde{\mu}$, etc. Note that $\tilde{\iota} = \iota_{n+1}$.

Let $\varepsilon > 0$, $\delta > 0$, and $f \in A_n$ be as in the hypotheses. Define

$$g(t) = \begin{pmatrix} y_1(t)\varphi^+(f_{1,1})(t)y_1(t)^* & y_1(t)\varphi^+(f_{1,2})(t)y_2(t)^* \\ y_2(t)\varphi^+(f_{2,1})(t)y_1(t)^* & y_2(t)\varphi^+(f_{2,2})(t)y_2(t)^* \end{pmatrix},$$

and note that

$$\tilde{y}(t)\tilde{\psi}(f)\tilde{y}(t)^* = \text{diag}(g(t), g(1-t)).$$

Accordingly, it suffices to prove that if

$$\|x(s)f(s)x(s)^* - x(t)f(t)x(t)^*\| < \varepsilon$$

for all $s, t \in [0, 1]$ such that $|s - t| < \delta$, then

$$\|g(s) - g(t)\| < \varepsilon$$

for all $s, t \in [0, 1]$ such that $|s - t| < 2\delta$.

Let v be the permutation matrix

$$v = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

When one calculates $vg(t)v^*$ by substituting the formulas for $y_j(t)$ and φ^+ in the expression for $g(t)$, the factors $c(t)$ and $c(t)^*$ all cancel out, and the final answer is

$$vg(t)v^* = \text{diag}\left(x\left(\frac{1}{2}t\right)f\left(\frac{1}{2}t\right)x\left(\frac{1}{2}t\right)^*, x\left(\frac{1}{2}(t+1)\right)f\left(\frac{1}{2}(t+1)\right)x\left(\frac{1}{2}(t+1)\right)^*\right).$$

Since we are assuming

$$\|x(s)f(s)x(s)^* - x(t)f(t)x(t)^*\| < \varepsilon$$

for all $s, t \in [0, 1]$ such that $|s - t| < \delta$, it is immediate that $|s - t| < 2\delta$ implies

$$\|vg(s)v^* - vg(t)v^*\| < \varepsilon,$$

whence also

$$\|g(s) - g(t)\| < \varepsilon,$$

as desired. ■

Proposition 12.14. The automorphism α of Example 12.11 generates an action of \mathbf{Z}_2 with the tracial Rokhlin property.

Proof. Let the notation be as before Lemma 12.12. Let τ be the unique tracial state on $A = \varinjlim A_n$. Define a tracial state τ_n on A_n by

$$\tau_n(f) = \int_0^1 \text{tr}_{4^n}(f(t)) dt,$$

where tr_m is the normalized trace on M_m . Then one checks that $\tau_{n+1} \circ \psi_n = \tau_n$ for all n . It follows from the uniqueness of τ that $\tau|_{A_n} = \tau_n$ for all n .

We use Theorem 8.2 to verify the tracial Rokhlin property. So let $F \subset A$ be finite and let $\varepsilon > 0$. Choose m and a finite set $S_0 \subset A_m$ such that every element of F is within $\frac{1}{8}\varepsilon$ of an element of S_0 .

The set S_0 is a uniformly equicontinuous set of functions from $[0, 1]$ to M_{4^m} , so there is $\delta > 0$ such that whenever $s, t \in [0, 1]$ satisfy $|s - t| < \delta$, then

$$\|f(s) - f(t)\| < \frac{1}{8}\varepsilon$$

for all $t \in [0, 1]$ and all $f \in S_0$. Choose $n \in \mathbf{N}$ with $n \geq m$ and so large that $2^{n-m}\delta > 1$. Apply Lemma 12.13 a total of $n - m$ times, the first time with $x(t) = 1$ for all t , obtaining after the last application a continuous unitary path $t \mapsto z(t)$ in $C([0, 1], M_{4^n})$ such that $\mu_n(z(t)) = z(t)$ for every $t \in [0, 1]$. Replacing $z(t)$ by $z(0)^*z(t)$, we may clearly assume that $z(0) = 1$. Then, in particular,

$$\|z(t)^*f(0)z(t) - f(t)\| < \frac{1}{8}\varepsilon$$

for all $t \in [0, 1]$ and all $f \in S_0$. Recall that we identify $C(S^1, B)$ with the set of functions $f \in C([0, 1], B)$ such that $f(0) = f(1)$. Since the fixed point algebra $A_n^{\alpha_n} = C(S^1, M_{4^n})^{\alpha_n}$ is just $C(S^1, M_{4^n}^{\mu_n})$, and since $M_{4^n}^{\mu_n}$ is finite dimensional, there is an α_n -invariant unitary $y \in A_n$ such that $y(t) = z(t)$ for $t \in [0, 1 - \frac{1}{8}\varepsilon]$. Then for each $f \in S_0$, regarded as a subset of A_n , there exists $g \in A_n$ such that $\|y^*gy - f\| < \frac{1}{4}\varepsilon$ and $g(t) = f(0)$ for $t \in [0, 1 - \frac{1}{8}\varepsilon]$. Let S be the set of all elements g obtained in this way from elements of S_0 . In particular, for every $a \in F$ there is $g \in S$ such that $\|a - y^*gy\| < \frac{1}{2}\varepsilon$.

We claim that there are orthogonal positive elements $b_0, b_1 \in A_{n+1} \subset A$ such that $b_j g = g b_j$ for all $g \in S$, and such that

$$\alpha_{n+1}(b_0) = b_1, \quad \alpha_{n+1}(b_1) = b_0, \quad 0 \leq b_0, b_1 \leq 1, \quad \text{and} \quad 0 \leq \tau(1 - b_0 - b_1) < \varepsilon.$$

For this purpose, it suffices to use in place of ψ_n the unitarily equivalent homomorphism

$$\tilde{\psi} = w\psi_n(-)w^* : C(S^1, M_{4^n}) \rightarrow C(S^1, M_{4^{n+1}})$$

in the proof of Lemma 12.13 (called ω_n in the proof of Proposition 5.1.1 of [3]), and to use in place of α_{n+1} the automorphism

$$\tilde{\alpha} = \text{Ad}(w) \circ \alpha_{n+1} = \text{Ad}(\tilde{u}) = \text{Ad}(\text{diag}(1, -1, -1, 1)).$$

Note that this change does not require any change in the formula for the trace τ_{n+1} , and also does not affect the first part of the conclusion of Lemma 12.12. Accordingly, if

$$g = \begin{pmatrix} g_{1,1} & g_{1,2} \\ g_{2,1} & g_{2,2} \end{pmatrix} \in S$$

then $\tilde{\psi}(g) \in C(S^1, M_{4^{n+1}})$ satisfies

$$\tilde{\psi}(g)(t) = \begin{pmatrix} g_{1,1}(0) & g_{1,2}(0) & 0 & 0 \\ g_{2,1}(0) & g_{2,2}(0) & 0 & 0 \\ 0 & 0 & g_{1,1}(0) & g_{1,2}(0) \\ 0 & 0 & g_{2,1}(0) & g_{2,2}(0) \end{pmatrix} = \begin{pmatrix} g(0) & 0 \\ 0 & g(0) \end{pmatrix}$$

for $t \in [\frac{1}{4}\varepsilon, 1 - \frac{1}{4}\varepsilon]$.

Now set

$$p_0 = \frac{1}{2} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix} \quad \text{and} \quad p_1 = \frac{1}{2} \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix},$$

both in $M_4(M_{4^n})$. In 2×2 block form, we can write

$$\begin{aligned} \tilde{u} &= \begin{pmatrix} s & 0 \\ 0 & -s \end{pmatrix} \quad \text{with} \quad s = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \\ p_0 &= \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad \text{and} \quad p_1 = \frac{1}{2} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}. \end{aligned}$$

With these formulas, it is easy to check that p_0 and p_1 are projections with $p_0 + p_1 = 1$, that p_0 and p_1 commute with $\tilde{\psi}(g)(t)$ for every $g \in S$ and $t \in [\frac{1}{4}\varepsilon, 1 - \frac{1}{4}\varepsilon]$, and that $\text{Ad}(\tilde{u})$ exchanges p_0 and p_1 .

Now choose and fix a continuous function $h: [0, 1] \rightarrow [0, 1]$ such that $h(t) = 0$ for $t \notin [\frac{1}{4}\varepsilon, 1 - \frac{1}{4}\varepsilon]$ and $h(t) = 1$ for $t \in [\frac{1}{2}\varepsilon, 1 - \frac{1}{2}\varepsilon]$, and define $b_j(t) = h(t)p_j$ for $j = 0, 1$. Then b_0 and b_1 are positive elements with $b_0, b_1 \leq 1$, which commute with $\tilde{\psi}(g)$ for every $g \in S$, which satisfy $b_0b_1 = 0$, such that $\text{Ad}(\tilde{u})$ exchanges b_0 and b_1 , and such that $0 \leq \tau_{n+1}(1 - b_0 - b_1) < \varepsilon$. This proves the claim above.

We return to the use of ψ_{n+1} , and we let $b_0, b_1 \in A_{n+1} \subset A$ be as in the claim (rather than its proof). In $A_{n+1} \subset A$, define $a_0 = y^*b_0y$ and $a_1 = y^*b_1y$. Since $\alpha(y) = y$, it follows that $a_0, a_1 \in A_{n+1} \subset A$ satisfy $a_j y^*gy = y^*gya_j$ for all $g \in S$, and

$$a_0a_1 = 0, \quad \alpha_{n+1}(a_0) = a_1, \quad 0 \leq a_0, a_1 \leq 1, \quad \text{and} \quad 0 \leq \tau(1 - a_0 - a_1) < \varepsilon.$$

For $a \in F$ choose $g \in S$ such that $\|a - y^*gy\| < \frac{1}{4}\varepsilon$. Then

$$\|[a_j, a]\| \leq 2\|a - y^*gy\| + \|[a_j, y^*gy]\| < 2(\frac{1}{2}\varepsilon) + 0 = \varepsilon.$$

This completes the verification of the hypotheses of Theorem 8.2, so it follows that α has the tracial Rokhlin property. ■

Proposition 12.15. Let $\alpha \in \text{Aut}(A)$ be as in Example 12.11. Then:

- (1) The action of \mathbf{Z}_2 generated by α does not have the strict Rokhlin property.
- (2) The dual action on $C^*(\mathbf{Z}_2, A, \alpha)$ has the strict Rokhlin property.
- (3) The generating automorphism $\hat{\alpha}$ of the dual action is tracially approximately inner.
- (4) The automorphism $\hat{\alpha}$ acts nontrivially on $K_1(C^*(\mathbf{Z}_2, A, \alpha))$.

Proof. We have already observed in Example 12.11 that $C^*(\mathbf{Z}_2, A, \alpha)$ is not AF. Therefore (1) follows from Theorem 11.2.

It is immediate from the discussion following Example 12.11 that the action of \mathbf{Z}_2 generated by α is approximately representable in the sense of Definition 3.6(2) of [29]. Part (2) therefore follows from Lemma 3.8(2) of [29].

We get (3) from Theorem 4.3. The algebra A has cancellation of projections because it is AF. It follows from Proposition 12.14 and Theorem 2.7 that $C^*(\mathbf{Z}_2, A, \alpha)$ has tracial rank zero. So projections in $C^*(\mathbf{Z}_2, A, \alpha)$ have the weak divisibility property (Definition 4.1) by Lemma 4.2. Therefore Theorem 4.3 applies.

It remains to prove (4). We continue to follow the notation introduced after Example 12.11. Let B_n be the fixed point algebra

$$A_n^{\alpha_n} = C(S^1, M_{2^{2n-1}}) \oplus C(S^1, M_{2^{2n-1}}) \subset C(S^1, M_{4^n}),$$

with the embedding being as 2×2 block diagonal matrices. Let $B = \varinjlim B_n$, which is also equal to A^α . Let $\beta_n \in \text{Aut}(B_n)$ be $\beta_n(f, g) = (g, f)$. By Proposition 5.2.2 of [3] and the preceding discussion, there is a corresponding automorphism β of the direct limit, $A \cong C^*(\mathbf{Z}_2, B, \beta)$, and the isomorphism can be chosen so that α generates the dual action. By Takai duality, it therefore suffices to show that β is nontrivial on $K_1(B)$.

Following the discussion after Corollary 5.3.2 of [3], let $v \in B_1 \subset A_1 = C(S^1, M_4)$ be the unitary

$$v(t) = \text{diag}(e^{2\pi it}, e^{2\pi it}, e^{-2\pi it}, e^{-2\pi it}).$$

As there, the image of $[v]$ in $K_1(B)$ is nonzero. It follows from Proposition 5.3.1 of [3] that $K_1(B)$ is torsion free, and one checks that $[\beta_1(v)] = -[v]$, so $\beta_*([v]) = -[v] \neq [v]$. ■

13. QUESTIONS

In this section, we state some open questions. The first one is suggested by attempts to weaken the hypotheses of Theorem 4.3. As far as we know, even the version for AF algebras and the strict Rokhlin property is open, so we state that as our second question.

Question 13.1. Let α be an action of \mathbf{Z}_n on a unital C^* -algebra A . Suppose that both A and $C^*(\mathbf{Z}_n, A, \alpha)$ are simple and have tracial rank zero, and that the automorphisms α_g and $\hat{\alpha}_\tau$ are all tracially approximately inner. Does it follow that α has the tracial Rokhlin property?

Question 13.2. Let α be an action of \mathbf{Z}_n on a unital C^* -algebra A . Suppose that both A and $C^*(\mathbf{Z}_n, A, \alpha)$ are simple and AF, and that the automorphisms α_g and $\hat{\alpha}_\tau$ are all approximately inner. Does it follow that α has the Rokhlin property?

The tracial Rokhlin property, as we have defined it, seems not to be useful much beyond the class of simple C^* -algebras with tracial rank zero. Therefore the following question seems important.

Question 13.3. What is the correct definition of the tracial Rokhlin property for actions of \mathbf{Z}_n on C^* -algebras which are not stably finite, which are stably finite but have badly behaved K-theory, which are not simple, or which have few or no nontrivial projections?

In particular, it seems of interest to have a suitable version of the tracial Rokhlin property for actions on purely infinite simple C^* -algebras, since results in Section 3 of [30] give strong restrictions on the K-theory of actions with the strict Rokhlin property.

Similarly, we want to modify the definition to make it suitable for actions on C^* -algebras which are not simple. One possibility is to add to Definition 1.1 a condition requiring that for a prespecified nonzero positive element a in the finite set F , one has $\|e_0 a e_0\| > \|a\| - \varepsilon$. Compare with Definition 2.1 of [37]. We do not know if this strengthening is adequate.

Another case of obvious interest is actions on C^* -algebras without a reasonable supply of projections. A test for any proposed definition is that it should imply our definition when there are in fact enough projections. This reasoning suggests the following question.

Question 13.4. Is there an analog of Lemma 8.1 or Theorem 8.2 for actions of \mathbf{Z}_n on simple unital C^* -algebras with tracial rank zero but more than one tracial state?

One might try simply writing some version of Definition 1.1 which uses positive elements in place of projections.

We next turn to tracial approximate innerness. Our first question is motivated by Lemmas 11.3 and 11.4.

Question 13.5. Let A be a simple separable unital nuclear C^* -algebra with tracial rank zero and satisfying the Universal Coefficient Theorem. Let $\alpha \in \text{Aut}(A)$. Suppose that α_* is the identity on the quotient of $K_0(A)$ by the infinitesimal subgroup $\text{Inf}(K_0(A))$. (Notation from [25].) Suppose further that for every $n \in \mathbb{N}$ there exists $\eta \in K_0(A)_+ \setminus \{0\}$ such that $\alpha_*(\eta) = \eta$ and $n\eta \leq [1_A]$. Does it follow that α is tracially approximately inner?

Since the definition of tracial approximate innerness suffers from the same difficulties as the definition of the tracial Rokhlin property, we also ask:

Question 13.6. What is the correct definition of tracial approximate innerness of automorphisms of C^* -algebras which are not stably finite, or which are stably finite but have badly behaved K-theory?

Work of H. Lin (in preparation) suggests that Question 13.5 has a positive answer, and that our definition is at least close to the correct one for automorphisms of simple unital C^* -algebras with tracial rank zero and satisfying the Universal Coefficient Theorem. The condition of approximate invariance of the projection, Condition (1) of Definition 3.1, may only be appropriate for automorphisms of finite order. Here also, one might try adding for more general algebras a condition such as $\|eae\| > \|a\| - \varepsilon$ for a prespecified nonzero positive element a in the finite set F .

One test for a good definition is that the composition of two tracially approximately inner automorphisms should again be tracially approximately inner. We don't know if this is true for our definition even on simple unital C^* -algebras with tracial rank zero. If we omit approximate invariance of e from the definition, then this should be true, and not hard to prove, whenever the order on projections over the algebra is determined by traces (Definition 1.8). Again, a potentially particularly interesting case is automorphisms of purely infinite simple C^* -algebras.

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